OSCILLATIONS

(SHM, DAMPED AND FORCED OSCILLATIONS & RESONANCE)

1. PERIODIC MOTION AND ITS CHARACTERISTICS AND TYPES OF SHM

1.1 Periodic Motion

(i) Any motion which repeats itself after regular interval of time is called periodic motion or harmonic motion.

(ii) The constant interval of time after which the motion is repeated is called time period.

Examples : (i) Motion of planets around the sun.

(ii) Motion of the pendulum of wall clock.

1.2 Oscillatory Motion

(i) The motion of a body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after regular interval of time.

(ii) The fixed point about which the body oscillates is called mean position or equilibrium position.

Examples : (i) Vibration of the wire of 'Sitar'.

(ii) Oscillation of the mass suspended from spring.

1.3 Simple Harmonic Motion

- In SHM particle does to and fro motion about a fixed point called mean position.
- The force acting on the particle which tends to bring the particle towards its mean position, is known as restoring force.
- This force is always directed towards the mean position and proportional to displacement from mean postion.
- SHM motion can be represented by

$$x = A\sin\omega t,$$
$$x = A\cos\omega t,$$
$$x = A\sin\omega t \pm B\cos\omega t,$$
$$x = A\sin^2\omega t,$$
$$x = A\cos^2(\omega t + \phi)$$

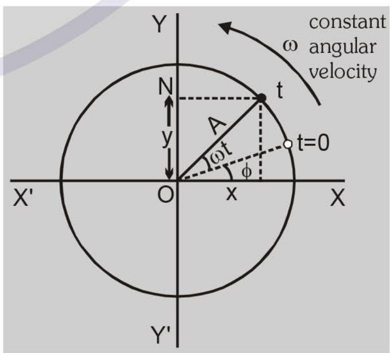
Amplitude

The maximum displacement of particle from its mean position is define as amplitude.

Time period (T), Frequency (n) and Angular frequency (ω)

- The time after which the particle keeps on repeating its motion is known as time period.
- It is given by $T = \frac{2\pi}{\omega}$, $T = \frac{1}{n}$ where ω is angular frequency and n is frequency.

Phase (SHM as a uniform circular motion)



(a) Projection of particle's position on Y-axis.

$y = A\sin(\omega t + \phi)$

The quantity $(\omega t + \phi)$ represents the phase angle at that instant.

(b) The phase angle at time t = 0 is known as **initial phase or epoch.**

(c) The difference of total phase angles of two particles executing S.H.M. with respect to the mean position is known as phase difference.

(d) If the phase angles of two particles executing S.H.M. are $(\omega t + \phi_1)$ and $(\omega t + \phi_2)$ respectively, then the phase difference between two particles is given by

$$\Delta\phi = (\omega t + \phi_2) - (\omega t + \phi_1) \quad \text{or} \quad \Delta\phi = \phi_2 - \phi_1$$

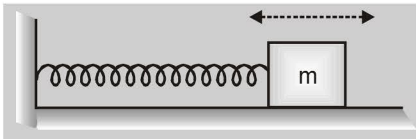
1.4 Types of Simple harmonic motion (S.H.M.)

(i) S.H.M. are of two types

(a) Linear S.H.M.

When a particle moves to and fro about a point (called equilibrium position) along a straight line then its motion is called linear simple harmonic motion.

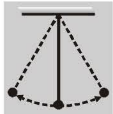
Example : Motion of a mass connected to spring.



(b) Angular S.H.M.

When a system oscillates angularly with respect to a axis then its motion is called angular simple harmonic motion.

Example :- Motion of a bob of simple pendulum.



(ii) Comparison between linear and angular S.H.M.

Linear S.H.M.	Angular S.H.M.
$F \propto -x$ $F = -kx$ Where k is the restoring force constant Where x is disp. from mean position	$\tau \propto -\theta$ $\tau = -C\theta$ Where C is the restoring torque constant Where θ is angular disp. from mean position.
$a = -\frac{k}{m}x$ whee m is the mass of body.	$\alpha = -\frac{C}{I}\theta$
$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$ It is known as differential equation of linear S.H.M.	$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$ It is known as differential equation of angular S.H.M.
$x = A \sin\omega t$ considering initial phase = 0 $a = -\omega^2 x$ where ω is the angular frequency	$\theta = \theta_0 \sin \omega t$ considering initial phase zero. $\alpha = -\omega^2\theta$
$\omega^2 = \frac{k}{m}$	$\omega^2 = \frac{C}{I}$
$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T} = 2\pi n$ where T is time period and n is frequency	$\omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$
$T = 2\pi\sqrt{\frac{m}{k}}$	$T = 2\pi\sqrt{\frac{I}{C}}$
$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	$n = \frac{1}{2\pi}\sqrt{\frac{C}{I}}$
This concept is valid for all types of linear S.H.M.	This concept is valid for all types of angular S.H.M.

2. SIMPLE HARMONIC MOTION (SHM) AND ITS EQUATION; VELOCITY, ACCELERATION

2.1 Displacement, Velocity and Acceleration in S.H.M.

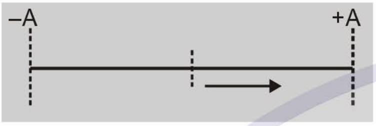
Displacement in S.H.M.

(i) The displacement of a particle executing linear S.H.M. at any instant is defined as the position of the particle from the mean position at that instant.

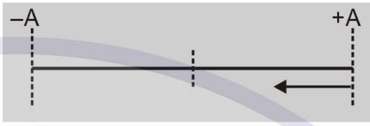
(ii) It can be given by relation $x = A \sin\omega t$ or $x = A \cos\omega t$ or $x = A \sin(\omega t + \phi)$

Ex. What will be the equation of displacement in the following different conditions ?

(i)



(ii)



(iii)

(iv)

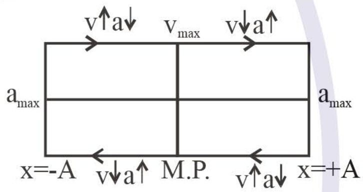
Sol. (i) $x = A \sin\omega t$

(ii) $x = A \sin(\omega t + \frac{\pi}{2}) \Rightarrow x = A \cos\omega t$

(iii) $x = A \sin(\omega t + \pi) \Rightarrow x = -A \sin\omega t$

(iv) $x = A \sin(\omega t + \frac{3\pi}{2}) \Rightarrow x = -A \cos\omega t$

2.2 Velocity in S.H.M.



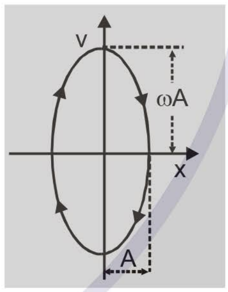
(i) Velocity in S.H.M. is given by

$$v = \frac{dx}{dt} = \frac{d}{dt}(A \sin \omega t)$$

$$\Rightarrow \quad v = A\omega \cos\omega t \Rightarrow \quad v = \pm\omega\sqrt{(A^2 - x^2)} \qquad v_{max} = A\omega$$

$$\text{OR} \quad \frac{v^2}{\omega^2 A^2} = 1 - \frac{x^2}{A^2} \quad \Rightarrow \quad \frac{x^2}{A^2} + \frac{v^2}{A^2\omega^2} = 1$$

This is equation of ellipse. So curve between displacement and velocity of particle executing S.H.M. is ellipse.



(ii) The graph between velocity and displacement is shown in figure.

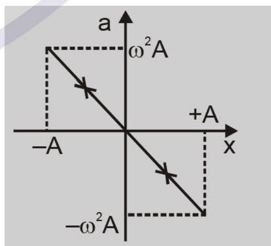
If particle oscillates with unit angular frequency ($\omega = 1$) then curve between v and x will be circle.

2.3 Acceleration in S.H.M.

(i) Acceleration in S.H.M. is given by $a = \frac{dv}{dt} = \frac{d}{dt}(A\omega \cos \omega t)$

$$a = -\omega^2 A \sin \omega t \quad \Rightarrow \quad a = -\omega^2 x$$

$$\Rightarrow \quad a_{max} = -\omega^2 A$$



(ii) The graph between acceleration and displacement is a straight line as shown in figure.

Important points :

- In linear S.H.M., the length of S.H.M. path = 2A
- In S.H.M., the total work done and displacement in one complete oscillation is zero but total travelled length is 4A.
- Velocity is always ahead of displacement by phase angle $\frac{\pi}{2}$ radian
- Acceleration is ahead of displacement by phase angle π radian i.e., opposite to displacement.
- Acceleration leads the velocity by phase angle $\frac{\pi}{2}$ radian.

3. ENERGY IN SHM – POTENTIAL & KINETIC ENERGIES

3.1 Potential Energy (U or P.E.)

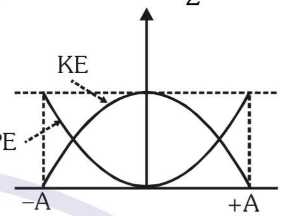
(i) In terms of displacement

The potential energy is related to conservative force by the relation $F = -\frac{dU}{dx} \Rightarrow \int dU = -\int F dx \Rightarrow U = \frac{1}{2}kx^2 + U_0$

Where the potenital energy at equilibrium position = U_0 If $U_0 = 0$ then $U = \frac{1}{2}kx^2$

(ii) In terms of time

Since $x = A\sin(\omega t + \phi)$, $U = \frac{1}{2}kA^2\sin^2(\omega t + \phi)$



$U_{max} = \frac{1}{2}m\omega^2A^2$

$KE_{max} = \frac{1}{2}m\omega^2A^2$

3.2 Kinetic Energy (K)

(i) In terms of displacement

If mass of the particle which is executing S.H.M. is m and Its velocity is v then kinetic energy at any instant.

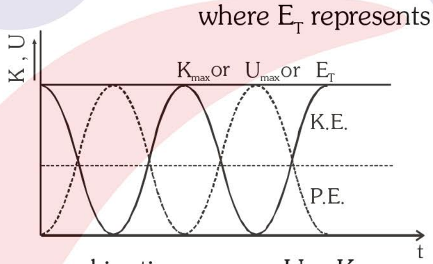
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

(ii) In terms of time

where E_T represents total energy

$\because v = A\omega\cos(\omega t + \phi)$

$\therefore K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$



3.3 Total energy (E)

Total energy in S.H.M. is given by ; E = potential energy + kinetic energy = U + K

w.r.t. position $E = \frac{1}{2}kx^2 + \frac{1}{2}k(A^2 - x^2) \Rightarrow E = \frac{1}{2}kA^2$ = constant

3.4 Average energy in S.H.M.

(a) $\langle KE \rangle_t = \frac{1}{4}m\omega^2A^2 = \frac{1}{4}kA^2$, $\langle KE \rangle_x = \frac{1}{3}m\omega^2A^2 = \frac{1}{3}kA^2$

(b) $\langle PE \rangle_t = \frac{1}{4}m\omega^2A^2 + U_0 = \frac{1}{4}kA^2 + U_0$, $\langle PE \rangle_x = \frac{1}{6}m\omega^2A^2 + U_0 = \frac{1}{6}kA^2 + U_0$

- The frequency of oscillation of potential energy and kinetic energy is twice as that of displacement or velocity or acceleration of a particle executing S.H.M.
- Frequency of total energy is zero because it remains constant.

4. OSCILLATIONS OF A SPRING BLOCK SYSTEM

4.1 Spring Block System

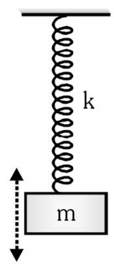
(i) When a small mass is suspended from a spring then this arrangement is known as spring block system.

For small linear displacement the motion of spring block system is simple harmonic.

(ii) For a spring block system

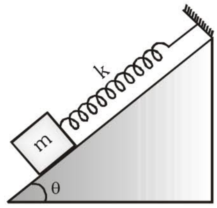
Time period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$,

Frequency $n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

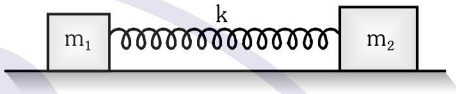


(iii) Time period of a spring block system is independent of acceleration due to gravity. This is why a clock based on oscillation of spring block system show proper time everywhere on a hill or moon or in a satellite or different places of earth, where gravity is varying.

(iv) If a spring block system oscillates in a vertical plane is made to oscillate on a horizontal surface or on an inclined plane then time period will remain unchanged.

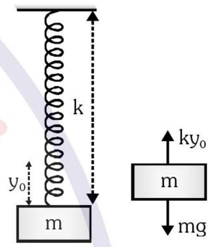


(v) If two masses m_1 and m_2 are connected by a spring and made to oscillate then time period $T = 2\pi\sqrt{\frac{\mu}{k}}$



Here, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ = reduced mass of a system.

(vi) If the stretch in a vertically loaded spring is y_0 then for equilibrium of mass m.



$$ky_0 = mg \quad \text{i.e.,} \quad \frac{m}{k} = \frac{y_0}{g}$$

So, time period $T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{y_0}{g}}$

But remember time period of spring pendulum is independent of acceleration due to gravity (y_0 will change with changing value of g).

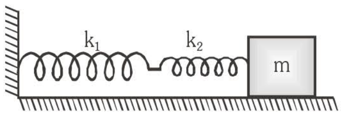
4.2 Various Spring Arrangements

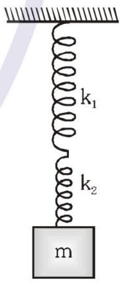
- **Series combination of springs**

In series combination same restoring force exerts in all springs but extension will be different.

If equivalent force constant is k_s then $F = -k_s x$

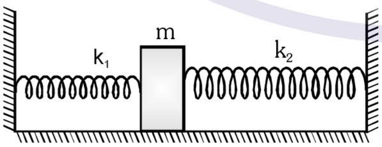
Where $\frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} \quad \Rightarrow \quad k_s = \frac{k_1 k_2}{k_1 + k_2}$

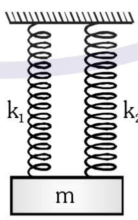


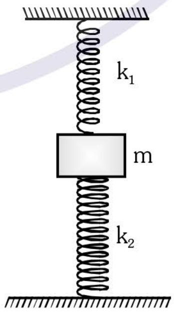


Time period $T_s = 2\pi\sqrt{\frac{m}{k_s}}$

- **Parallel Combination of springs**







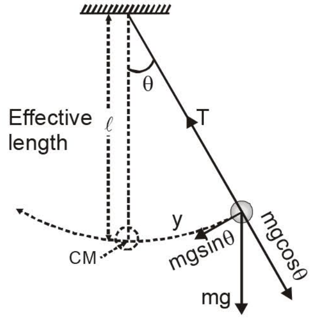
In parallel combination, displacement on each spring is same but restoring force is different.

If equivalent force constant is k_P, then $F = -k_P x$, so $k_P = k_1 + k_2$

Time period $T_P = 2\pi\sqrt{\frac{m}{k_P}} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$

5. SIMPLE PENDULUM

If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum



$$T = 2\pi\sqrt{\frac{\ell}{g}}$$

Important points :

1. The time period of pendulum is independent from mass of the bob but it depends on size of bob (position of centre of mass). So in pendulum when a solid iron bob is replaced by light aluminium bob of same radius then time period remains unchanged.

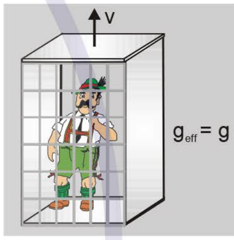
2. If simple pendulum is shifted to poles, equator or hilly areas, then its time period may be different $\left(T \propto \frac{1}{\sqrt{g}}\right)$

3. If a clock based on oscillation of simple pendulum is shifted from earth to moon then it becomes slow because its time period increases and becomes $\sqrt{6}$ times compare to earth. $\frac{g_M}{g_E} = \frac{1}{6} \Rightarrow T_M = \sqrt{6}T_E$

4. **Periodic time of simple pendulum in reference (system) frames.**

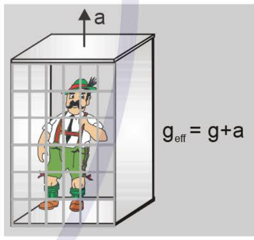
$$T = 2\pi\sqrt{\frac{\ell}{g_{eff}}}$$

(a) If reference system is lift



(i) If velocity of lift v = constant

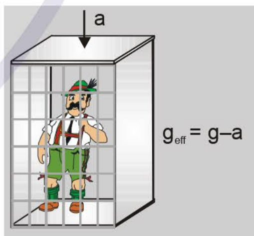
acceleration $a = 0$ and $g_{eff.} = g$ $\therefore$ $T = 2\pi\sqrt{\frac{\ell}{g}}$



(ii) If lift is moving upwards with acceleration a

$g_{eff.} = g + a$

$T = 2\pi\sqrt{\frac{\ell}{g+a}}$ $\Rightarrow$ T decreases



(iii) If lift is moving downwards with acceleration a

$g_{eff.} = g - a$

$\therefore$ $T = 2\pi\sqrt{\frac{\ell}{g-a}}$ $\Rightarrow$ T increases

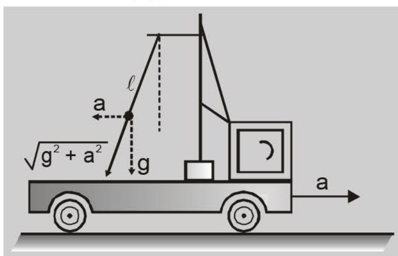
(iv) If lift falls downwards freely

$g_{eff.} = g - g = 0$ $\Rightarrow$ $T = \infty$ simple pendulum will not oscillate

(b) A simple pendulum is mounted on a moving truck

(i) If truck is moving with constant velocity, time period remains same $T = 2\pi\sqrt{\frac{\ell}{g}}$

(ii) If truck accelerates forward with acceleration 'a'.



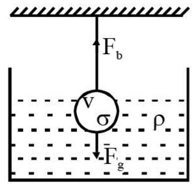
So effective acceleration, $g_{eff.} = \sqrt{g^2 + a^2}$ and $T' = 2\pi\sqrt{\frac{\ell}{g_{eff.}}}$

Time period $T' = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2+a^2}}}$ $\Rightarrow T'$ decreases

6. If a simple pendulum of density σ is made to oscillate in a liquid of density ρ then its time period will increase as compare to that of air and is given by

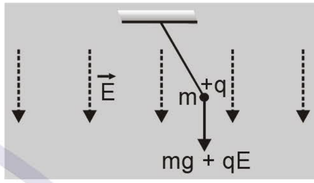
$$g_{net} = g\left(1 - \frac{\rho}{\sigma}\right)$$

$$T = 2\pi\sqrt{\frac{\ell}{\left[1 - \frac{\rho}{\sigma}\right]g}}$$



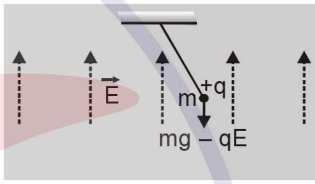
7.(a) If the bob of simple pendulum has **positive** charge q and pendulum is placed in uniform electric field which is in downward direction.

$$T = 2\pi\sqrt{\frac{\ell}{g + \frac{qE}{m}}}$$



(b) If the bob of simple pendulum has **positive** charge q and is made is oscillate in uniform electric field acting in upward direction.

$$T = 2\pi\sqrt{\frac{\ell}{g - \frac{qE}{m}}}$$



8. $T = 2\pi\sqrt{\frac{\ell}{g}}$ is valid when length of simple pendulum (ℓ) is negligible as compare to radius of earth ($\ell << R$) but if ℓ is comparable to radius of earth

then time period $T = 2\pi\sqrt{\frac{1}{\left[\frac{1}{\ell} + \frac{1}{R}\right]g}}$

The time period of oscillation of simple pendulum of infinite length

$T = 2\pi\sqrt{\frac{R}{g}} \simeq 84.6$ minute $\approx 1\frac{1}{2}$ hour It is maximum time period.

9. Second's pendulum

If the time period of a simple pendulum is 2 second then it is called second's pendulum. Second's pendulum take one second to go from one extreme position to other extreme position.

For second's pendulum, time period $T = 2 = 2\pi\sqrt{\frac{\ell}{g}}$

At the surface of earth $g = 9.8\ m/s^2 \approx \pi^2 m/s^2$,

So length of second pendulum at the surface of earth $\ell \approx 1$ m and at surface of moon $\ell \approx 1/6$ m

10. If simple pendulum is shifted to the centre of earth, freely falling lift or in artificial satellite then it will not oscillate and its time period is infinite ($\because g_{eff.} = 0$).

11. Variation in Time Period with Temperature

$T = 2\pi\sqrt{\frac{\ell_0(1+\alpha\Delta t)}{8}}$ where Δt is charge in temp.

or $\boxed{\frac{\Delta T}{T_0} = \frac{1}{2} \propto \Delta T}$ where ΔT increase in time period

If time period of clock based on simple pendulum increases then clock will be slow and if time period decreases then clock will be fast.

6. DIFFERENT TYPES OF OSCILLATIONS

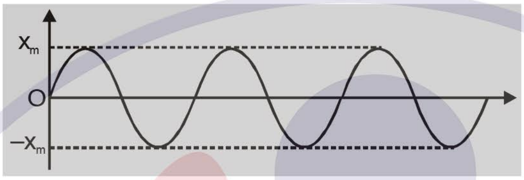
Different types of oscillations

Free, Damped, Forced oscillations and Resonance

(a) Free oscillation

(i) The oscillations of a particle with fundamental frequency under the influence of restoring force are defined as free oscillations.

(ii) The amplitude, frequency and energy of oscillations remain constant.



(b) Damped oscillations

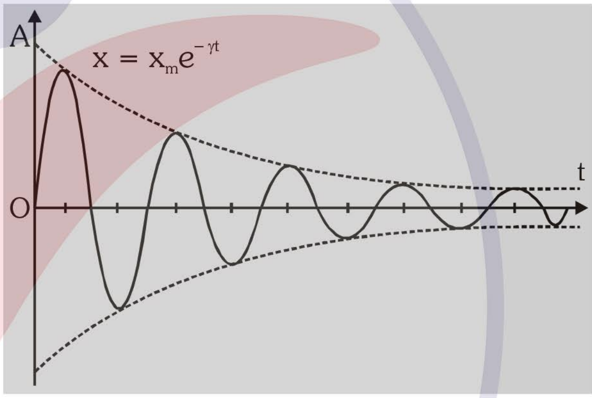
(i) In these oscillations the amplitude of oscillations decreases exponentially due to damping forces like frictional force, viscous force etc.

(ii) If initial amplitude is A_0 then amplitude after time t will be $A = A_0e^{-\gamma t}$ where

γ = Damping coefficient

For example A_0 is initial amplitude

$A_0 \xrightarrow{t_0} \frac{A_0}{n} \xrightarrow{t_0} \frac{A_0}{n^2}$ **and so on.**



(c) Forced oscillations

(i) The oscillations in which a body oscillates under the influence of an external periodic force (driver) are known as forced oscillations.

(ii) The driven body does not oscillate with its natural frequency rather it oscillates with the frequency of the driver.

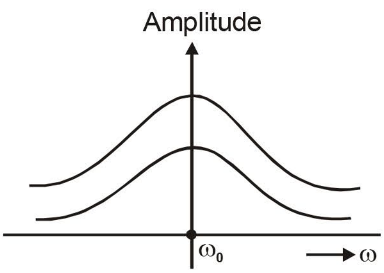
(iii) The amplitude of forced vibration is determined by the difference between the frequency of the applied force and the natural frequency. If the difference between frequencies is small then the amplitude will be large.

(d) Resonance

(i) When the frequency of external force (driver) ω is equal to the natural frequency ω_0 of the oscillator (driven), then this state of the driver and the driven is known as the state of resonance.

(ii) In the state of resonance, there occurs maximum transfer of energy from the driver to the driven.

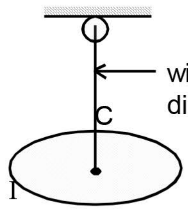
(iii) When $\omega = \omega_0$, amplitude of oscillations is maximum. This state is resonance. Energy of the oscillations is also maximum in this state. Amplitude vs ω graph is as shown.



7. EXAMPLES OF SIMPLE HARMONIC MOTION

PHYSICAL SYSTEM THAT MOVES WITH SHM

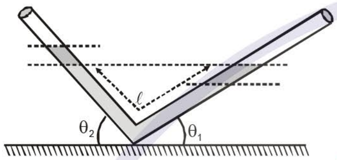
(1) Torsional oscillator (Angular SHM)



$$T = 2\pi\sqrt{\frac{I}{C}}, \qquad C = \frac{\eta\pi r^4}{2\ell}$$

η = modulas of elasticity of the wire
r = radius of the wire
ℓ = length of the wire
I = MI of the disc

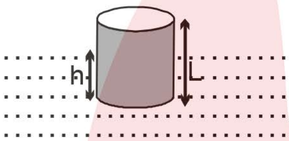
(2) Oscillation liquid in a V-shape tube :



$$T = 2\pi\sqrt{\frac{\ell}{g(\cos\theta_1 + \cos\theta_2)}}$$

ℓ = Total length of liquid column in tubes
θ_1 & θ_2 = angles of tubes with horizontal
(for U-shaped tube $\theta_1 = \theta_2 = 90°$)
g = gravitational acceleration

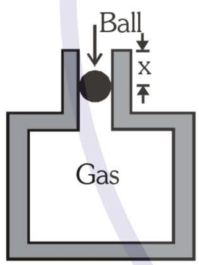
(3) Floating block :

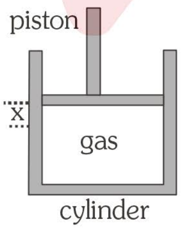


$$T = 2\pi\sqrt{\frac{m}{A\rho g}}, \quad T = 2\pi\sqrt{\frac{Ld}{\rho g}}, \quad T = 2\pi\sqrt{\frac{h}{g}}$$

m = mass of block, A = Area of block
ρ = density of liquid, L = length of cylinder
d = density of cylinder, h = length of cylinder inside the liquid at mean position

(4) Oscillation of piston in a gas chamber piston:





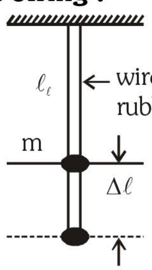
$$T = 2\pi\sqrt{\frac{Vm}{A^2K}}$$

v = volume of cylinder, m = mass of piston
A = area of cylinder ball

$$K = \text{bulk modulus} = \frac{\Delta P}{-\Delta V / V}$$

For (1) Isothernal process K=P, (2) adiabatic process K = γP

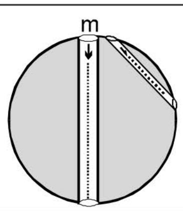
(5) Longitudinal oscillation of an elastic wire or rubber string :



$$T = 2\pi\sqrt{\frac{\ell m}{AY}}$$

ℓ = length of string
m = mass of ball
A = Area of cross section
Y = young's modulus

(6) Tunnel across earth :



$$T = 2\pi\sqrt{\frac{R_e}{g}}, \qquad R_e = 6400 \text{ km}, \quad T = 84.6 \text{ minute}$$

Time taken to go from one end of the tunnel to other end is T/2 i.e. 42.3 minutes

Illustration An object performs S.H.M. of amplitude 5 cm and time period 4 s. If timing is started when the object is at the centre of the oscillation i.e., x = 0 then calculate –

(i) Frequency of oscillation

(ii) The displacement at 0.5 sec.

(iii) The maximum acceleration of the object.

(iv) The velocity at a displacement of 3 cm.

Solution (i) Frequency $f = \frac{1}{T} = \frac{1}{4} = 0.25$ Hz

(ii) The displacement equation of object $x = A\sin\omega t$

so at t = 0.5 s $x = 5\sin(2\pi \times 0.25 \times 0.5) = 5\sin\frac{\pi}{4} = \frac{5}{\sqrt{2}}$ cm

(iii) Maximum acceleration $a_{max} = \omega^2 A = (0.5\,\pi)^2 \times 5 = 12.3$ cm/s^2

(iv) Velocity at x = 3 cm is $v = \pm\omega\sqrt{A^2 - x^2} = \pm 0.5\pi\sqrt{5^2 - 3^2} = \pm 6.28$ cm/s

Illustration Amplitude of a harmonic oscillator is A, when velocity of particle is half of maximum velocity, then determine position of particle.

Solution $v = \omega\sqrt{A^2 - x^2}$ but $v = \frac{v_{max}}{2} = \frac{A\omega}{2}$

$\frac{A\omega}{2} = \omega\sqrt{A^2 - x^2}$ $\Rightarrow$ $A^2 = 4[A^2 - x^2]$

$\Rightarrow$ $x^2 = \frac{4A^2 - A^2}{4}$ $\Rightarrow$ $x = \pm\frac{\sqrt{3}A}{2}$

Illustration A particle performing SHM is found at its equilibrium position at t = 1 sec and it is found to have a speed of 0.25 m/s at t = 2 sec. If the period of oscillation is 8 sec. Calculate the amplitude of oscillations.

Solution $x = A\sin(\omega t + \phi)$

at t = 1 sec. particle at mean position

$$0 = A\sin\left(\frac{2\pi}{8} \times 1 + \phi\right) \Rightarrow \boxed{\phi = -\frac{\pi}{4}}$$

at t = 2 sec. velocity of particle is 0.25 m/s

$$0.25 = A\omega\cos\left(\frac{\pi}{4} \times 2 - \frac{\pi}{4}\right)$$

$$0.25 = \frac{A\omega}{\sqrt{2}} \Rightarrow \boxed{A = \frac{\sqrt{2}}{\pi}}$$

Illustration If two S.H.M. are represented by equations $y_1 = 10\sin\left[3\pi t + \frac{\pi}{4}\right]$ and $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right]$ then find the ratio of their amplitudes and phase difference in between them.

Solution As $y_2 = 5\left[\sin(3\pi t) + \sqrt{3}\cos(3\pi t)\right] = 10\left[\frac{1}{2}\sin(3\pi t) + \frac{\sqrt{3}}{2}\cos(3\pi t)\right]$

$\Rightarrow$ $10\left[\cos\frac{\pi}{3}\sin 3\pi t + \sin\frac{\pi}{3}\cos 3\pi t\right] = 10\sin\left(3\pi t + \frac{\pi}{3}\right)$ $\Rightarrow$ $\frac{A_1}{A_2} = \frac{10}{10}$

$\Rightarrow$ $A_1 : A_2 = 1 : 1$ and Phase difference $= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$ rad.

Illustration The velocity of a particle in S.H.M. at position x_1 and x_2 are v_1 and v_2 respectively. Determine value of time period and amplitude.

Solution $v = \omega\sqrt{A^2 - x^2}$ $\Rightarrow$ $v^2 = \omega^2 (A^2 - x^2)$

At position x_1 velocity $v_1^2 = \omega^2 (A^2 - x_1^2)$ ... (i)

At position x_2 velocity $v_2^2 = \omega^2 (A^2 - x_2^2)$ ... (ii)

Subtracting (ii) from (i) $v_1^2 - v_2^2 = \omega^2(x_2^2 - x_1^2)$ $\Rightarrow$ $\omega = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}}$

Time period $T = \frac{2\pi}{\omega}$ $\Rightarrow$ $T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$

Dividing (i) by (ii) $\frac{v_1^2}{v_2^2} = \frac{A^2 - x_1^2}{A^2 - x_2^2}$ $\Rightarrow$ $v_1^2 A^2 - v_1^2 x_2^2 = v_2^2 A^2 - v_2^2 x_1^2$

So $A^2(v_1^2 - v_2^2) = v_1^2 x_2^2 - v_2^2 x_1^2$ $\Rightarrow$ $A = \sqrt{\frac{v_1^2 x_2^2 - v_2^2 x_1^2}{v_1^2 - v_2^2}}$

Illustration In case of simple harmonic motion –

(a) What fraction of total energy is kinetic and what fraction is potential when displacement is one half of the amplitude.

(b) At what displacement the kinetic and potential energies are equal.

Solution In S.H.M.

$KE = \frac{1}{2}k(A^2 - x^2)$ $PE = \frac{1}{2}kx^2$ and $TE = \frac{1}{2}kA^2$

(a) $f_{KE} = \frac{KE}{TE} = \frac{A^2 - x^2}{A^2}$ $f_{PE} = \frac{PE}{TE} = \frac{x^2}{A^2}$

at $x = \frac{A}{2}$ $f_{KE} = \frac{A^2 - A^2/4}{A^2} = \frac{3}{4}$ and $f_{PE} = \frac{A^2/4}{A^2} = \frac{1}{4}$

(b) KE = PE $\Rightarrow$ $\frac{1}{2}k(A^2 - x^2) = \frac{1}{2}kx^2$ $\Rightarrow$ $2x^2 = A^2$ $\Rightarrow$ $x = \pm\frac{A}{\sqrt{2}}$

Illustration A particle starts oscillating simple harmonically from its equilibrium position with time period T. Determine ratio of K.E. and P.E. of the particle at time $t = \frac{T}{12}$.

Solution at $t = \frac{T}{12}$ $x = A\sin\frac{2\pi}{T}\times\frac{T}{12} = A\sin\frac{\pi}{6} = \frac{A}{2}$

so $K.E. = \frac{1}{2}k(A^2 - x^2) = \frac{3}{4}\times\frac{1}{2}kA^2$ and $P.E. = \frac{1}{2}kx^2 = \frac{1}{4}\times\frac{1}{2}kA^2$ $\therefore$ $\frac{K.E.}{P.E.} = \frac{3}{1}$

Illustration The potential energy of a particle oscillating on x-axis is $U = 20 + (x-2)^2$. Here U is in joules and x in meters. Total mechanical energy of the particle is 36 J.

(a) State whether the motion of the particle is simple harmonic or not ?

(b) Find the mean position.

(c) Find the maximum kinetic energy of the particle.

Solution (a) $F = -\frac{dU}{dx} = -2(x-2)$ By assuming $x - 2 = X$, we have $F = -2X$

Since, $F \propto -X$ The motion of the particle is simple harmonic

(b) The mean position of the particle is $X = 0 \Rightarrow x - 2 = 0$, which gives $x = 2m$

(c) Maximum kinetic energy of the particle is, $K_{max} = E - U_{min} = 36 - 20 = 16$ J

Note : U_{min} is 20 J at mean position or at $x = 2m$.

Illustration A body of mass m attached to a spring which is oscillating with time period 4 seconds. If the mass of the body is increased by 4 kg, its time period increases by 2 sec. Determine value of initial mass m.

Solution In I^{st} *case* : $T = 2\pi\sqrt{\frac{m}{k}} \Rightarrow 4 = 2\pi\sqrt{\frac{m}{k}}$ ...(i) and in II^{nd} *case:* $6 = 2\pi\sqrt{\frac{m+4}{k}}$ ...(ii)

Divide (i) by (ii) $\frac{4}{6} = \sqrt{\frac{m}{m+4}} \Rightarrow \frac{16}{36} = \frac{m}{m+4} \Rightarrow m = 3.2$ kg

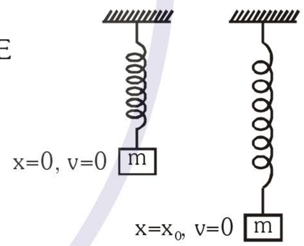
Illustration One body is suspended from a spring of length ℓ, spring constant k and has time period T. Now if spring is divided in two equal parts which are joined in parallel and the same body is suspended from this arrangement then determine new time period.

Solution Spring constant in parallel combination $k' = 2k + 2k = 4k$

$$\therefore\ T' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{m}{4k}} = 2\pi\sqrt{\frac{m}{k}} \times \frac{1}{\sqrt{4}} = \frac{T}{\sqrt{4}} = \frac{T}{2}$$

Illustration A block of mass m is attached from a spring of spring constant k and dropped from its natural length. Find the amplitude of S.H.M.

Solution Let amplitude of S.H.M. be A then by work energy theorem $W = \Delta KE$

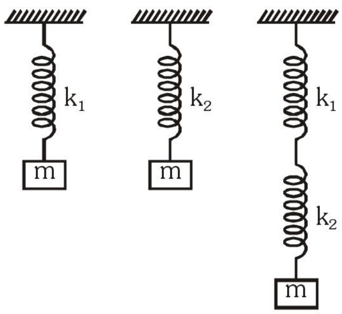


$$mgx_0 - \frac{1}{2}kx_0^2 = 0 \Rightarrow x_0 = \frac{2mg}{k}$$

So amplitude $A = \frac{mg}{k}$

Illustration Periodic time of oscillation T_1 is obtained when a mass is suspended from a spring. If another spring is used with same mass then periodic time of oscillation is T_2. Now if this mass is suspended from series combination of above springs then calculate the time period.

Solution $T_1 = 2\pi\sqrt{\frac{m}{k_1}} \Rightarrow k_1 = \frac{4\pi^2 m}{T_1^2}$ and $T_2 = 2\pi\sqrt{\frac{m}{k_2}} \Rightarrow k_2 = \frac{4\pi^2 m}{T_2^2}$ so $K_{eq.} = \frac{4\pi^2 m}{T_{eq.}^2}$



In series combination $\frac{1}{K_{eq.}} = \frac{1}{K_1} + \frac{1}{K_2} \Rightarrow \frac{T_{eq.}^2}{4\pi^2 m} = \frac{T_1^2}{4\pi^2 m} + \frac{T_2^2}{4\pi^2 m} \Rightarrow T_{eq.} = \sqrt{T_1^2 + T_2^2}$

Illustration Infinite springs with force constants k, 2k, 4k, 8k, ... respectively are connected in series. Calculate the effective force constant of the spring.

Solution

$$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \ldots\ldots\ldots\ldots\infty$$

(For infinite G.P. $S_\infty = \frac{a}{1-r}$ where a = First term, r = common ratio)

$$\frac{1}{k_{eff}} = \frac{1}{k}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots\right] = \frac{1}{k}\left[\frac{1}{1-\frac{1}{2}}\right] = \frac{2}{k} \quad \text{so } k_{eff} = k/2$$

Illustration A simple pendulum is suspended from the ceiling of a lift. When the lift is at rest, its time period is T. With what acceleration should lift be accelerated upwards in order to reduce its time period to $\frac{T}{2}$?

Solution In stationary lift $T = 2\pi\sqrt{\frac{\ell}{g}}$ ...(i) In accelerated lift $\frac{T}{2} = T' = 2\pi\sqrt{\frac{\ell}{g+a}}$ ...(ii)

$$\Rightarrow \quad 2 = \sqrt{\frac{g+a}{g}} \quad \Rightarrow \quad g + a = 4g \quad \Rightarrow \quad a = 3g$$

13. If length of a simple pendulum is increased by 4%. Then determine percentage change in time period.

Solution

$$T = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow T \propto \ell^{1/2} \Rightarrow \frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta \ell}{\ell}$$

Percentage change in time period $\frac{\Delta T}{T} \times 100\% = \frac{1}{2}\frac{\Delta\ell}{\ell} \times 100$ $[\because \Delta g = 0]$

According to question $\frac{\Delta\ell}{\ell} \times 100 = 4\%$ $\therefore$ $\frac{\Delta T}{T} \times 100\% = \frac{1}{2} \times 4\% = 2\%$

Illustration A bob of simple pendulum is suspended by a metalic wire. If α is the coefficient of linear expansion and $d\theta$ is the change in temperature then prove that percentage change in time period is $50\alpha d\theta$.

Solution With change in temperature $d\theta$, the effective length of wire becomes $\ell' = \ell(1 + \alpha\, d\theta)$

$T' = 2\pi\sqrt{\frac{\ell'}{g}}$ and $T = 2\pi\sqrt{\frac{\ell}{g}}$ Hence $\frac{T'}{T} = \sqrt{\frac{\ell'}{\ell}} = (1 + \alpha\, d\theta)^{1/2} = 1 + \frac{1}{2}\alpha d\theta$

$$\therefore \quad \text{Percentage increase in time period} = \left[\frac{T' - T}{T}\right] \times 100 = \left[\frac{T'}{T} - 1\right] \times 100$$

$$= \left[1 + \frac{\alpha d\theta}{2} - 1\right] \times 100 = 50\,\alpha\, d\theta$$

Illustration The amplitude of a damped oscillator becomes half in one minute. The amplitude after 3 minutes will be $\frac{1}{x}$ times of the original. Determine the value of x.

Solution Amplitude of damped oscillation is $A = A_0 e^{-\gamma t}$ [from $x = x_m e^{-\gamma t}$]

at t = 1 min $A = \frac{A_0}{2}$ so $\frac{A_0}{2} = A_0 e^{-\gamma}$ $\Rightarrow$ $e^\gamma = 2$

After 3 minutes $A = \frac{A_0}{x}$ so $\frac{A_0}{x} = A_0 e^{-\gamma \times 3}$ $\Rightarrow$ $x = e^{3\gamma} = (e^\gamma)^3 = 2^3 = 8$