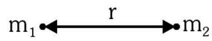
GRAVITATION

- **Newton's law of gravitation**



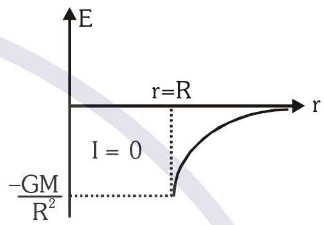
Force of attraction between two point masses $F = \frac{Gm_1m_2}{r^2}$

Directed along the line joining of point masses.

• It is a conservative force field ⇒ mechanical energy is conserved.

• It is a central force field ⇒ angular momentum is conserved.

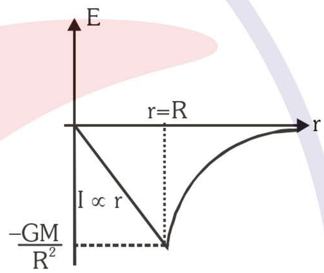
- **Gravitational field due to spherical shell**



- Outside the shell $E_g = \frac{GM}{r^2}$, where r > R
- On the surface $E_g = \frac{GM}{R^2}$, where r=R
- Inside the shell $E_g = 0$, where r<R

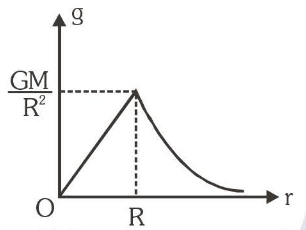
[Note : Direction always towards the centre of the sphere]

- **Gravitational field due to solid sphere**



- Outside the sphere $E_g = \frac{GM}{r^2}$, where r > R
- On the surface $E_g = \frac{GM}{R^2}$, where r=R
- Inside the sphere $E_g = \frac{GMr}{R^3}$, where r< R

- **Acceleration due to gravity** $g = \frac{GM}{R^2}$ (at surface)



(Taking direction towards centre of earth as positive)

- At height h above surface of earth,

  $g_h = \frac{GM}{(R+h)^2}$ If h << R ; $g_h \approx g_s\left(1-\frac{2h}{R}\right)$

- At depth d below the surface of earth, $g_d = \frac{GM(R-d)}{R^3} = g_s\left(1-\frac{d}{R}\right)$

- Effect of rotation on g

  $g' = g - \omega^2 R\cos^2\lambda$ where λ is angle of latitude.

Condition of weightlessness on Earth's surface

If apparent weight of body is zero then angular speed of Earth can be calculated as $mg' = mg - mR_e\omega^2\cos^2\lambda$

$$0 = mg - mR_e\omega^2\cos^2\lambda \Rightarrow \omega = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R_e}}$$

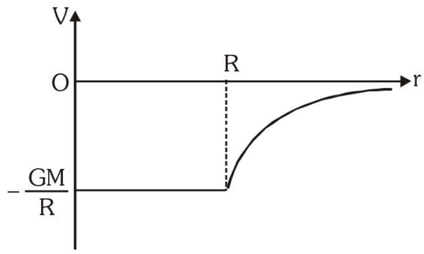
But at equator $\lambda = 0°$ $\therefore \omega = \sqrt{\frac{g}{R_e}} = \frac{1}{800}$ rad/s = 0.00125 rad/s = 1.25×10^{-3} rad/s.

Note : If Earth were to rotate with 17 times of its present angular speed then bodies lying on equator would fly off into the space. Time period of Earth's rotation in this case would be 1.4 h.

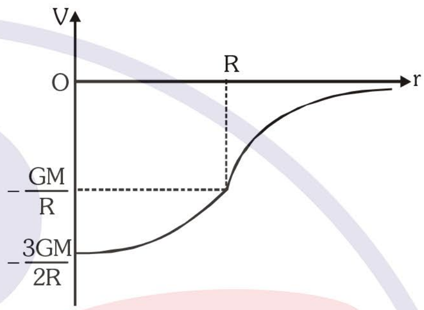
- **Gravitational potential**

Due to a point mass at a distance $V = -\frac{GM}{r}$

- **Gravitational potential due to spherical shell**
  - Outside the shell $V = -\frac{GM}{r}$, $r>R$
  - Inside/on the surface the shell $V = -\frac{GM}{R}$, $r \leq R$

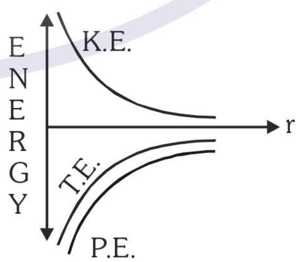


- **Potential due to solid sphere**
  - Outside the sphere $V = -\frac{GM}{r}$, $r > R$
  - On the surface $V = -\frac{GM}{R}$, $r = R$
  - Inside the sphere $V = -\frac{GM(3R^2 - r^2)}{2R^3}$, $r < R$



- **Potential on the axis of a thin ring at a distance x from centre** $V = -\frac{GM}{\sqrt{R^2 + x^2}}$
- **Escape velocity from a planet of mass M and radius R** $v_e = \sqrt{\frac{2GM}{R}}$
- **Orbital velocity of satellite** $v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R+h)}}$
  - For nearby satellite $v_0 = \sqrt{\frac{GM}{R}} = \frac{v_e}{\sqrt{2}}$

    Here v_e = escape velocity
- $v_e = \sqrt{2}v_0$, for near by setellite v_0 = 8 km/s
- **Time period of satellite** $T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM}}$
- **Energies of a satellite**
  - Potential energy $U = -\frac{GMm}{r}$
  - Kinetic energy $K = \frac{1}{2}mv^2 = \frac{GMm}{2r}$
  - Mechanical energy $E = U + K = -\frac{GMm}{2r}$
  - Binding energy $BE = -E = \frac{GMm}{2r}$



- **Kepler's laws**
  - **I[st] Law :** Law of orbit

    Path of a planet is elliptical with the sun at a focus.
  - **II[nd] Law :** Law of area : Areal velocity $\frac{dA}{dt}$ = constant $= \frac{L}{2m}$
  - **III[rd] Law :** Law of periods $T^2 \propto a^3$ or $T^2 \propto \left(\frac{r_{max} + r_{min}}{2}\right)^3 \propto (\text{mean radius})^3$

    For circular orbits $T^2 \propto R^3$

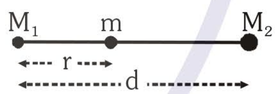
KEY POINTS

- At the centre of earth, a body has centre of mass, but no centre of gravity.
- The centre of mass and centre of gravity of a body coincide if gravitation field is uniform.
- You do not experience gravitational force in daily life due to objects of same size as value of G is very small.
- Moon travellers tie heavy weight at their back before landing on Moon due to smaller value of g at Moon.
- Space rockets are usually launched in equatorial line from West to East because g is minimum at equator and earth rotates from West to East about its axis.
- Angular momentum in gravitational field is conserved because gravitational force is a central force.
- Kepler's second law or constancy of areal velocity is a consequence of conservation of angular momentum.

Illustration Two stationary particles of masses M_1 and M_2 are 'd' distance apart. A third particle lying on the line joining the particles, experiences no resultant gravitational force. What is the distance of this particle from M_1?

Solution Let m be the mass of the third particle

Force on m towards M_1 is $F_1 = \frac{GM_1m}{r^2}$



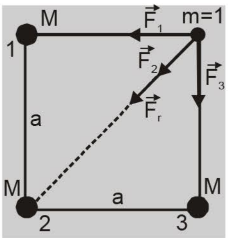
Force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

Since net force on m is zero $\therefore F_1 = F_2$

$$\Rightarrow \frac{GM_1m}{r^2} = \frac{GM_2m}{(d-r)^2} \Rightarrow \left(\frac{d-r}{r}\right)^2 = \frac{M_2}{M_1} \Rightarrow \frac{d}{r} - 1 = \frac{\sqrt{M_2}}{\sqrt{M_1}} \Rightarrow r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}\right]$$

Illustration Three masses, each equal to M are placed at the three corners of a square of side a. Calculate the force of attraction on unit mass placed at the fourth corner.

Solution Force on m = 1 due to masses at corners 1 and 3 are $\vec{F_1}$ and $\vec{F_3}$ with $F_1 = F_3 = \frac{GM}{a^2}$ resultant of $\vec{F_1}$ and $\vec{F_3}$ is $F_r = \sqrt{2}\frac{GM}{a^2}$ and its direction is along the diagonal i.e. toward corner 2



Force on m due to mass M at 2 is $F_2 = \frac{GM}{(\sqrt{2}a)^2} = \frac{GM}{2a^2}$; F_r and F_2 act in the same direction.

Resultant of these two is the net force :

$$F_{net} = \frac{\sqrt{2}GM}{a^2} + \frac{GM}{2a^2} = \frac{GM}{a^2}\left[\sqrt{2} + \frac{1}{2}\right]$$; it is directed along the diagonal as shown in the figure.

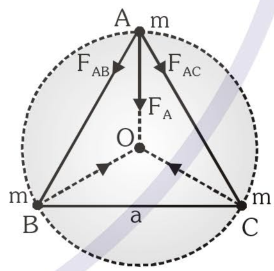
Illustration Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side 'a'. The only forces acting on the particles are their mutual gravitational forces. It is intended that each particle moves along a circle while maintaining their original separation 'a'. Determine the initial velocity that should be given to each particle and the time period of the circular motion.

Solution The resultant force on particle at A due to other two particles is

$$F_A = \sqrt{F_{AB}^2 + F_{AC}^2 + 2F_{AB}F_{AC}\cos 60^\circ} = \sqrt{3}\frac{Gm^2}{a^2} \quad ...(i) \qquad \left[\because F_{AB} = F_{AC} = \frac{Gm^2}{a^2}\right]$$

Radius of the circle $r = \frac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v, so that the resultant force acts as the centripetal force,

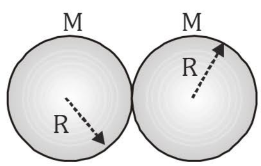


then $$\frac{mv^2}{r} = \sqrt{3}\frac{mv^2}{a} \quad ...(ii)$$

From (i) and (ii) , $$\sqrt{3}\frac{mv^2}{a} = \frac{Gm^2\sqrt{3}}{a^2} \Rightarrow v = \sqrt{\frac{Gm}{a}}$$

Time period $$T = \frac{2\pi r}{v} = \frac{2\pi a}{\sqrt{3}}\sqrt{\frac{a}{Gm}} = 2\pi\sqrt{\frac{a^3}{3Gm}}.$$

Illustration Two solid spheres of same size of a certain metal are placed in contact with each other. Prove that the gravitational force acting between them is directly proportional to the fourth power of their radius.



Solution The weights of the spheres may be assumed to be concentrated at their centres.

So $$F = \frac{G\left[\frac{4}{3}\pi R^3\rho\right] \times \left[\frac{4}{3}\pi R^3\rho\right]}{(2R)^2} = \frac{4}{9}(G\pi^2\rho^2)R^4 \quad \therefore \quad F \propto R^4$$

Illustration A body of mass m is placed on the surface of earth. Find the work required to lift this body by a height

(i) $h = \frac{R_e}{1000}$ (ii) $h = R_e$

Solution (i) $h = \frac{R_e}{1000}$, as $h << R_e$, so

we can apply $W_{ext} = mgh$; $W_{ext} = (m)\left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{1000}\right) = \frac{GM_e m}{1000R_e}$

(ii) $h = R_e$, in this case h is not very less than R_e, so we cannot apply $\Delta U = mgh$

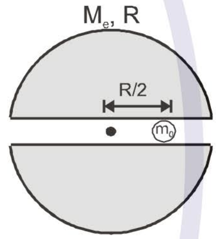
$$W_{ext} = U_f - U_i = m(V_f - V_i) \; ; \; W_{ext} = m\left[\left(-\frac{GM_e}{R_e + R_e}\right) - \left(-\frac{GM_e}{R_e}\right)\right] \; ; \; W_{ext} = \frac{GM_e m}{2R_e}.$$

Illustration If velocity given to an object from the surface of the Earth is n times the escape velocity then what will be its residual velocity at infinity ?

Solution Let the residual velocity be v, then from energy conservation $\frac{1}{2}m(nv_e)^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 + 0$

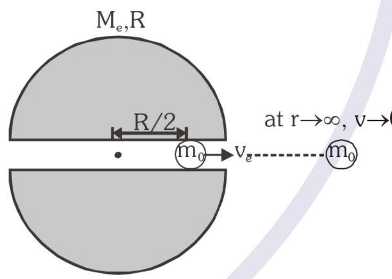
$$\Rightarrow v^2 = n^2v_e^2 - \frac{2GM}{R} = n^2v_e^2 - v_e^2 = (n^2 - 1)\,v_e^2 \Rightarrow v = \left(\sqrt{n^2 - 1}\right)v_e.$$

Illustration A narrow tunnel is dug along the diameter of the earth, and a particle of mass m_0 is placed at $\frac{R}{2}$ distance from the centre. Find the escape speed of the particle from that place.



Solution Suppose we project the particle with speed v_e, so that it just reaches infinity ($r \to \infty$).

Applying energy conservation principle



$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}m_0v_e^2 + m_0\left[-\frac{GM_e}{2R^3}\left\{3R^2 - \left(\frac{R}{2}\right)^2\right\}\right] = 0$$

$$\Rightarrow v_e = \sqrt{\frac{11GM_e}{4R}}.$$

Illustration A particle is projected vertically upwards from the surface of the earth (radius R_e) with a speed equal to one fourth of escape velocity. What is the maximum height attained by it ?

Solution From conservation of mechanical energy, $\frac{1}{2}mv^2 = \frac{GMm}{R_e} - \frac{GMm}{R}$

Where R = maximum distance from centre of the earth Also $v = \frac{1}{4}v_e = \frac{1}{4}\sqrt{\frac{2GM}{R_e}}$

$$\Rightarrow \frac{1}{2}m \times \frac{1}{16} \times \frac{2GM}{R_e} = \frac{GMm}{R_e} - \frac{GMm}{R} \Rightarrow R = \frac{16}{15}R_e \Rightarrow h = R - R_e = \frac{R_e}{15}.$$

Illustration Gravitational potential difference between a point on the surface of a planet and point 10 m above is 4 J/kg. Considering the gravitational field to be uniform, how much work is done in moving a mass of 2 kg from the surface to a point 5 m above the surface?

Solution Gravitational field $g = -\frac{\Delta V}{\Delta x} = -\left(\frac{-4}{10}\right) = \frac{4}{10}\ J/kg-m$

Work done in moving a mass of 2 kg from the surface to a point 5 m above the surface,

$$W = mgh = (2\ kg)\left(\frac{4}{10}\frac{J}{kg-m}\right)(5\ m) = 4\ J$$

Illustration Two satellites S_1 and S_2 are revolving round a planet in coplanar and concentric circular orbits of radii R_1 and R_2 in the same sense respectively. Their respective periods of revolution are 1 h and 8 h. The radius of the orbit of satellite S_1 is equal to 10^4 km. Find the relative speed in km/h when they are closest.

Solution By Kepler's 3[rd] law, $\frac{T^2}{R^3}$ = constant $\therefore \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3}$ or $\frac{1}{(10^4)^3} = \frac{64}{R_2^3}$ or $R_2 = 4 \times 10^4$ km

Distance travelled in one revolution, $S_1 = 2\pi R_1 = 2\pi \times 10^4$ and $S_2 = 2\pi R_2 = 2\pi \times 4 \times 10^4$

$$v_1 = \frac{S_1}{t_1} = \frac{2\pi \times 10^4}{1} = 2\pi \times 10^4\ km/h \text{ and } v_2 = \frac{S_2}{t_2} = \frac{2\pi \times 4 \times 10^4}{8} = \pi \times 10^4\ km/h$$

$\therefore$ Relative velocity = $v_1 - v_2 = 2\pi \times 10^4 - \pi \times 10^4 = \pi \times 10^4$ km/h

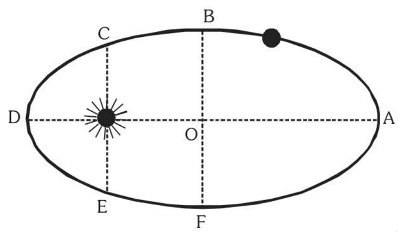
Illustration A planet is revolving round the sun in an elliptical orbit as shown in figure. Select correct alternative(s)

(A) Its total energy is negative at D.

(B) Its angular momentum is constant

(C) Net torque on the planet about sun is zero

(D) Linear momentum of the planet is conserved



Solution **(Ans. A, B, C)**

For (A) : For a bound system, the total energy is always negative.

For (B) : For central force field, angular momentum is always conserved.

For (C) : For central force field, torque = 0.

For (D) : In presence of external force, linear momentum is not conserved.