ROTATIONAL MOTION

RIGID BODY :

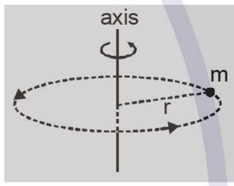
Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time) that means the shape and size do not change, during the motion. Eg : Fan, Pen, Table, stone and so on.

Type of Motion of rigid body

- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

ROTATIONAL MOTION

Moment of Inertia

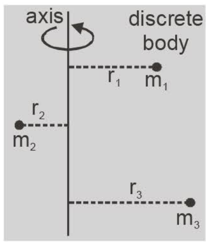


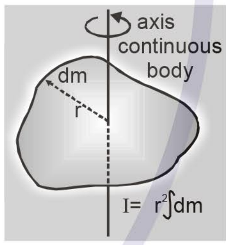
The virtue by which a body revolving about an axis opposes the change in rotational motion is known as moment of inertia.

• The moment of inertia of a particles with respect to an axis of rotation is equal to the product of mass of the particle and square of distance from rotational axis. $I = mr^2$

r = perpendicular distance from axis of rotation

• Moment of inertia of system of particle





$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...$$

• **For Rigid Bodies :**

Moment of inertia of a rigid body about any axis of rotation.

$$I = \int dm\, r^2$$

Radius of Gyration (K)

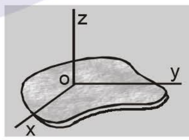
$$I = MK^2$$

K has no meaning without axis of rotation. K is a scalar quantity

Radius of gyaration $K = \sqrt{\dfrac{I}{M}}$

Perpendicular axis Theorems : $I_z = I_x + I_y$

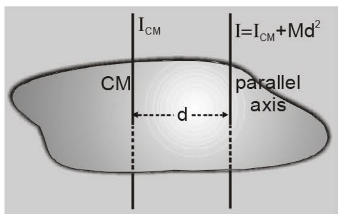
(Valid only for 2-dimensional body)



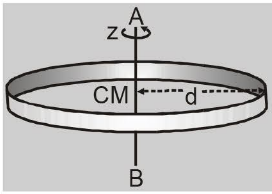
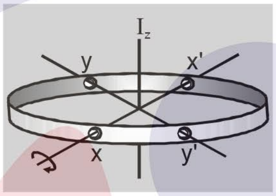
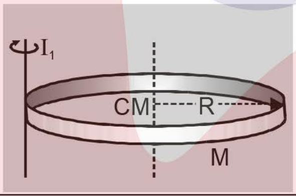
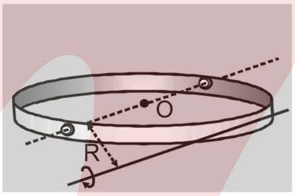
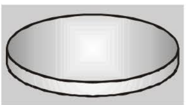
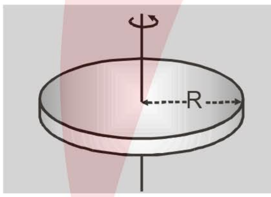
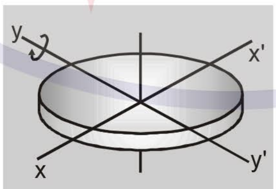
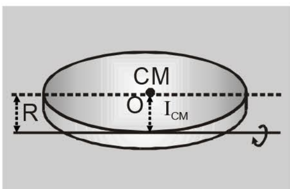
Parallel axis Theorem : $I = I_{CM} + Md^2$

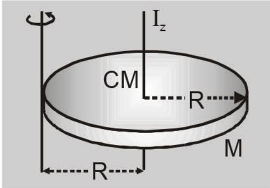
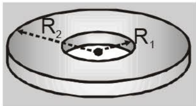
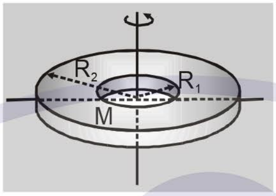
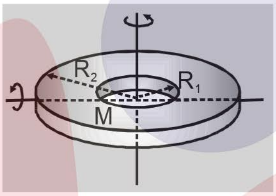
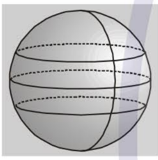
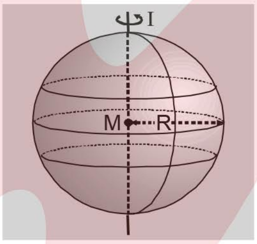
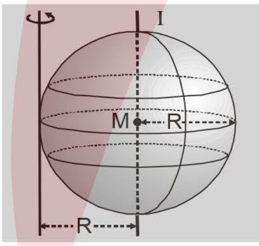
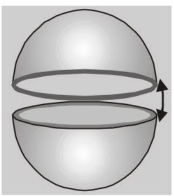
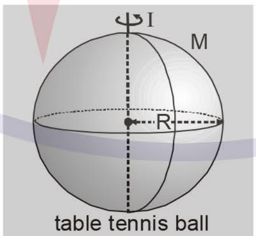
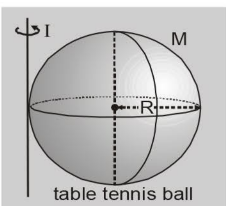
(for all type of bodies)

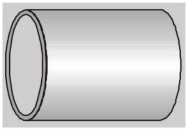
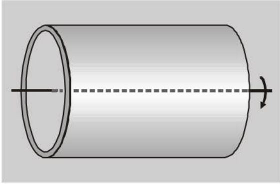
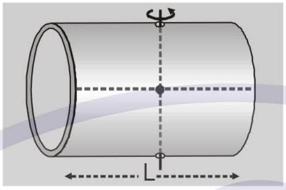
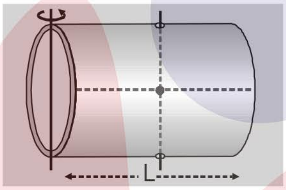
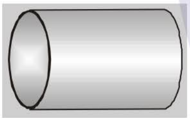
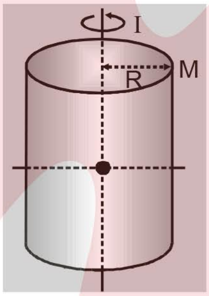
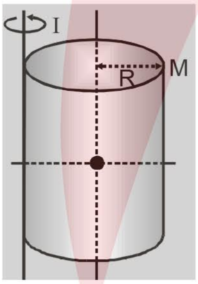
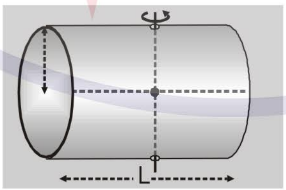
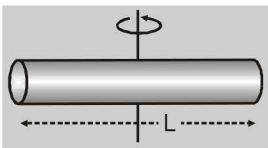
I_{CM} = moment of inertia about the axis
Passing through the centre of mass

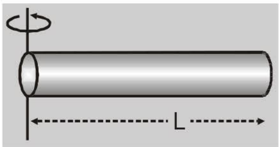
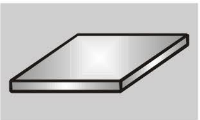
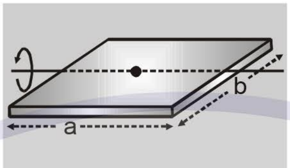
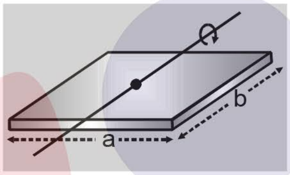
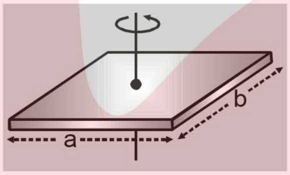
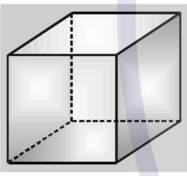
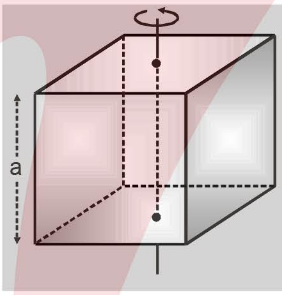


MOMENT OF INERTIA OF SOME REGULAR BODIES

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Circular Ring  Mass = M	(a) About an axis perpendicular to the plane and passes through the centre		MR^2	R
Radius = R	(b) About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c) About an axis tangential to the rim and perpendicular to the plane of the ring		$2MR^2$	$\sqrt{2}\,R$
	(d) About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}\,R$
(2) Circular Disc  M = Mass R = Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b) About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c) About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$

	(d) About an axis tangential to the rim & perpendicular to the plane of disc		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) About a diameteric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
(5) Hollow Sphere (Thin spherical Shell)  M = Mass R = Radius Thickness negligible	(a) About diametric axis passing through centre of mass		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
	(b) About a tangent to the surface		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$

(6) Hollow Cylinder  M = Mass R = Radius L = Length	(a) About its geometrical axis which is parallel to its length		MR^2	R
	(b) About an axis which is perpendicular to its length and passes through its centre of mass		$\frac{MR^2}{2}+\frac{ML^2}{12}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c) About an axis perpendicular to its length and passing through one end of the cylinder		$\frac{MR^2}{2}+\frac{ML^2}{3}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
(7) Solid Cylinder M = Mass R = Radius L = Length 	(a) About its geometrical axis, which is along its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) About an axis tangential to the cylinderical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c) About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}+\frac{MR^2}{4}$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$
(8) Thin Rod 	(a) About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$

Thickness is negligible w.r.t. length Mass = M Length = L	(b) About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(9) Rectangular Plate 	(a) About an axis passing through centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
M = Mass a = Length b = Breadth	(b) About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c) About an axis passing throught centre of mass and perpendicular to plane		$\frac{M(a^2+b^2)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$
(10) Cube  Mass = M Side a	About an axis passes through centre of mass and perpendicular to face		$\frac{Ma^2}{6}$	$\frac{a}{\sqrt{6}}$

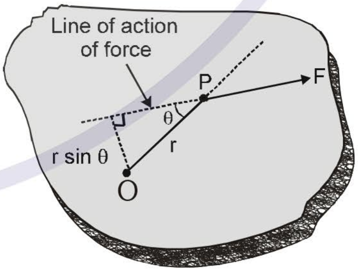
TORQUE

Torque about point : $\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude of torque = Force × perpendicular distance of line of action of force from the axis of rotation.

$\tau = r F \sin\theta$

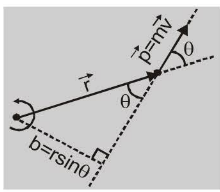
Direction of torque can be determined by using right hand thumb rule.



ANGULAR MOMENTUM (MOMENT OF LINEAR MOMENTUM)

Angular momentum of a body about a given axis is the product of its linear momentum and perpendicular distance of line of action of linear momentum vector from the axis of rotation

$$\vec{L} = \vec{r} \times \vec{p}$$



Magnitude of Angular momentum = Linear momentum × Perpendicular distance of line of action of momentum from the axis of rotation

$L = mv \times r \sin\theta$

Direction of angular momentum can be used by using right hand thumb rule.

- According to Newtons Second Law's for rotatry motion $\vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$.
- Angular Impulse = Change in angular momentum.
- If a large torque acts on a body for a small time then, angular impulse = $\vec{\tau}dt$

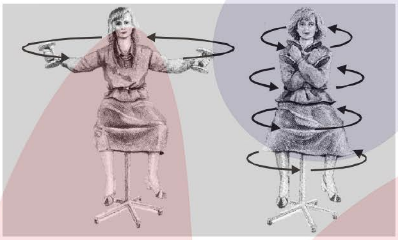
Conservation of Angular Momentum

Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

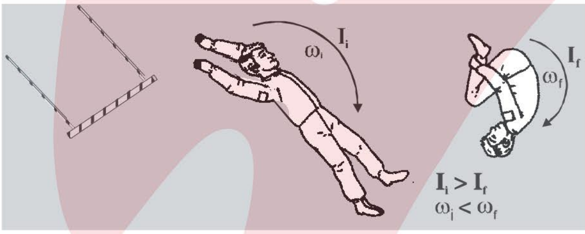
If $\tau = 0$ then $\frac{\Delta L}{\Delta t} = 0 \Rightarrow L = \text{constant} \Rightarrow L_f = L_i$ or $I_1\omega_1 = I_2\omega_2$

Examples

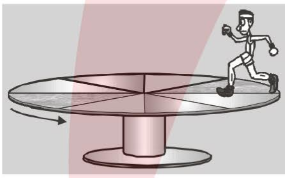
- If a person skating on ice folds his arms then his M.I. decreases and 'ω' increases.



- A diver jumping from a height folds his arms and legs (I decrease) in order to increase no. of rotation in air by increasing 'ω'.



- If a person moves towards the centre of rotating platform then 'I' decrease and 'ω' increase.



ROTATIONAL KINETIC ENERGY

Kinetic Energy of Rotation

$$KE_R = \frac{1}{2}I\omega^2$$

- other forms $K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I} = \frac{1}{2}L\omega$
- If external torque acting on a body is equal to zero ($\tau = 0$), L=constant $K \propto \frac{1}{I}$, $K \propto \omega$
- Rotational Work $W_r = \tau\theta$ (If torque is constant)

  $W_r = \int_{\theta_1}^{\theta_2} \tau d\theta$ (If torque is variable)

- The work done by torque = Change in kinetic energy of rotation.

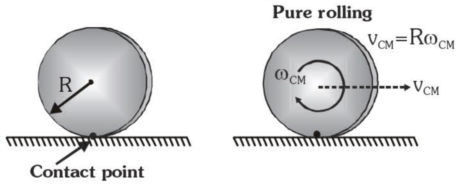
$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$$

- Instantaneous power $P_{in} = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$ Average power $P_{av} = \frac{w}{t}$

COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

When a body perform translatory motion as well as rotatory motion then it is known as rolling.

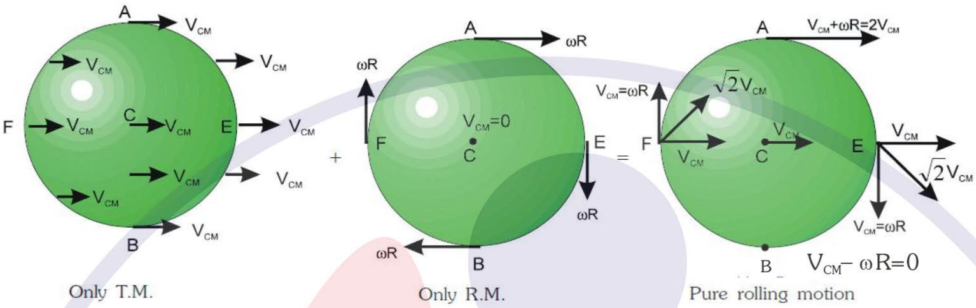
In Pure Rolling



(i) If the velocity of point of contact with respect to the surface is zero then it is known as pure rolling.

(ii) If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by

$$\vec{v} = \vec{v}_{CM} + \vec{\omega}_{CM} \times \vec{R}$$

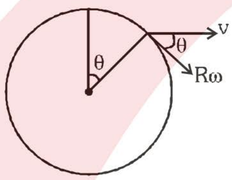


Only Translatory motion + Only Rotatory Motion = Rolling motion. For pure rolling above body

$V_A = 2V_{CM}$ $\qquad$ $V_E = \sqrt{2}\ V_{CM}$

$V_F = \sqrt{2}\ V_{CM}$ $\qquad$ $V_B = 0$

Velocity at a point on rim of sphere

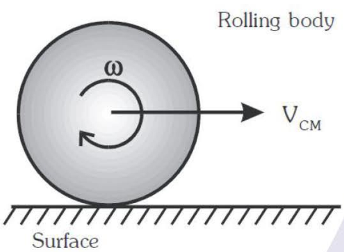


$$v_{net} = \sqrt{v^2 + R^2\omega^2 + 2vR\omega\cos\theta}$$

For pure rolling $v = R\omega$

$$\boxed{v_{net} = 2v\cos\frac{\theta}{2}}$$

Rolling Kinetic Energy under pure rolling



Rolling Kinenetic Energy $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$

Rolling Kinenetic Energy $E = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right)$

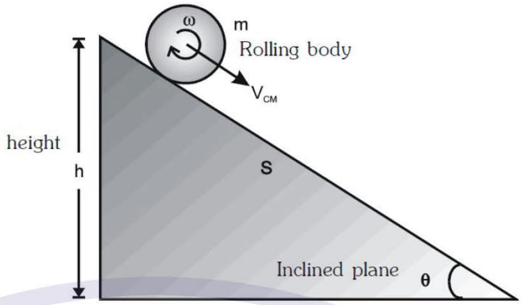
$E_{translation} : E_{rotation} : E_{Total} = 1 : \frac{K^2}{R^2} : 1+\frac{K^2}{R^2}$

Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\frac{K^2}{R^2}}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{1+\frac{K^2}{R^2}}$	$\frac{E_{rotation}}{E_{total}} = \frac{K^2/R^2}{1+\frac{K^2}{R^2}}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Rolling Motion on an inclined plane

Velocity at bottom of inclined plane

Applying Conservation of energy



$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad ...(1)$$

$$h = s\sin\theta \quad ...(2)$$

from (1) & (2)

$$V_{Rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

- Linear accleration on reaching the lowest point $a = \frac{g\sin\theta}{1+K^2/R^2}$
- Time taken to reach the lowest point of the plane is $t = \sqrt{\frac{2s(1+K^2/R^2)}{g\sin\theta}}$
- $\frac{K^2}{R^2}$ Least, will reach first

  $\frac{K^2}{R^2}$ Maximum, will reach last

  $\frac{K^2}{R^2}$ equal, will reach together
- When ring, disc, hollows sphere, solid sphere rolls on same inclined plane then

  $v_S > v_D > v_H > v_R$ $\quad$ $a_S > a_D > a_H > a_R$ $\quad$ $t_S < t_D < t_H < t_R$

Illustration A wheel is rotating with angular velocity 2 rad/s. It is subjected to a uniform angular acceleration 2.0 rad/s^2.

(a) What angular velocity does the wheel acquire after 10 s?

(b) How many revolutions will it make in this time interval?

Solution. The wheel is in uniform angular acceleration, Hence –

$\omega = \omega_o + \alpha t \rightarrow$ Substituting the values of ω_o, α and t, we have

$\omega = 2 + 2\times 10 = 22$ rad/s

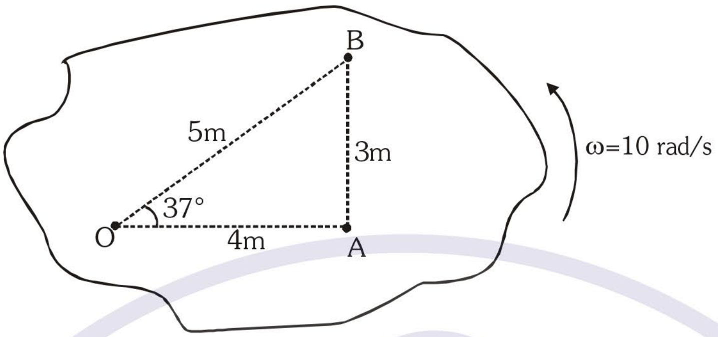
$\theta = \theta_o + \frac{1}{2}(\omega_o + \omega)t \rightarrow$ Substituting $\theta_o = 0$ for initial position, and ω_o from above equation, we have

$$\theta = 0 + \tfrac{1}{2}(2+22)10 = 120\,\text{rad.}$$

In one revolution, the wheel rotates through 2π radians. Therefore the number of complete revolutions n is

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} \approx 19$$

Illustration A rigid lamina is rotating about an axis passing perpendiuclar to its plane through point O as shown in the figure.

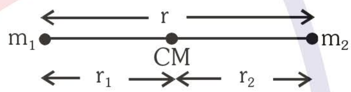


The angular velocity of point B w.r.t. A is

Solution In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

Illustration Two masses m_1 and m_2 are placed at a separation r. Find out the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the line joining the masses.

Solution. $m_1r_1 = m_2r_2$ and $r_1 + r_2 = r \Rightarrow r_1 = \frac{m_2 r}{m_1+m_2}$, $r_2 = \frac{m_1 r}{m_1+m_2}$

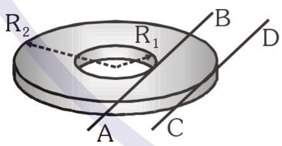


Moment of inertia $I = m_1r_1^2 + m_2r_2^2 = m_1\left(\frac{m_2 r}{m_1+m_2}\right)^2 + m_2\left(\frac{m_1 r}{m_1+m_2}\right)^2 = \left(\frac{m_1 m_2}{m_1+m_2}\right)r^2$

Note : Here $I = \mu r^2$ where μ (reduced mass) $= \frac{m_1 m_2}{m_1+m_2}$.

Illustration Calculate the moment of inertia of an annular disc about an axis which lies in the plane of the disc and tangential to the (i) inner circle and (ii) outer circle. Mass of the disc is M and its inner radius is R_1 and outer radius is R_2.

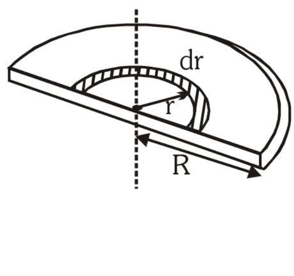
Solution. (i) M.I. about an axis tangential to the inner circle is $I_{AB} = \frac{M}{4}(R_1^2+R_2^2) + MR_1^2$



(ii) M.I. about an axis tangential to the outer circle is $I_{CD} = \frac{M}{4}(R_1^2+R_2^2) + MR_2^2$

Illustration Calculate the moment of Inertia of a semicircular disc of mass M and radius R about an axis passing through its centre and perpendicular to its plane.

Solution Let us assume a ring of radius 'r' & thickness 'dr'

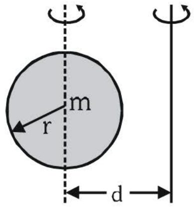


$$dm = \frac{M}{\frac{\pi R^2}{2}}(\pi r dr) = \frac{2Mrdr}{R^2}$$

$$I = \int r^2 dm = \int_0^R r^2 \frac{2Mr}{R^2} dr = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R \Rightarrow I = \frac{MR^2}{2}$$

Illustration The radius of gyration of a solid sphere of radius r about a certain axis is r. Calculate the distance of that axis from the centre of the sphere.

Solution. From paralle axes theorem,



$$\because \quad I = I_{CM} + md^2 \quad \therefore \; mr^2 = \frac{2}{5}mr^2 + md^2$$

$$\Rightarrow \quad d = \sqrt{\frac{3}{5}}r = \sqrt{0.6}\, r.$$

Illustration Find the moment of inertia of the ring shown in figure about the axis AB.

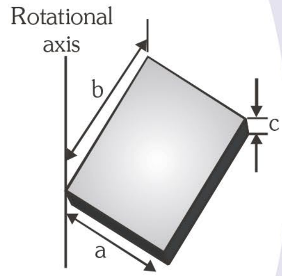
Solution. From parallel axes theorem,

$$I_{AB} = I_{CM} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

Illustration The uniform solid block shown in figure has mass M and edge dimensions a, b, and c. Calculate its rotational inertia about an axis passing through one corner and perpendicular to the large faces.

Solution. Use the parallel - axes theorem. The rotational inertia of a rectangular slab about an axis through the centre and perpendicular to the large face is given by

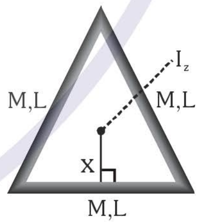
$$I_{cm} = \frac{M}{12}\left(a^2 + b^2\right)$$



A parallel axis through a corner is at distance $h = \sqrt{(a/2)^2 + (b/2)^2}$

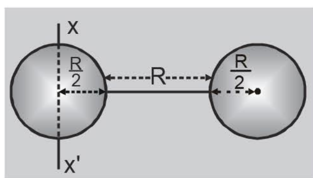
from the centre, so $I = I_{cm} + Mh^2 = \frac{M}{12}(a^2 + b^2) + \frac{M}{4}(a^2 + b^2) = \frac{M}{3}(a^2 + b^2).$

Illustration Three rods are arranged in the form of an equilateral triangle. Calculate the M.I. about an axis passing through the geometrical centre and perpendicular to the plane of the triangle (Assume that mass and length of each rod is M and L respectively).



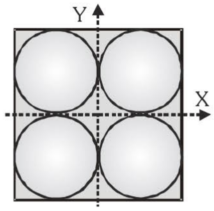
Solution. $$I = 3I_{CM} + 3M\,x^2 = \frac{3ML^2}{12} + 3M\left(\frac{L}{2\sqrt{3}}\right)^2 = \frac{ML^2}{4} + \frac{ML^2}{4} = \frac{ML^2}{2}.$$

Illustration Diameter of each spherical shell is R and mass M they are joined by a light and massless rod. Calculate the moment of inertia of the system about xx' axis.



Solution. $$I_{system} = \frac{2}{3}M\left(\frac{R}{2}\right)^2 + \left[\frac{2}{3}M\left(\frac{R}{2}\right)^2 + M(2R)^2\right] = \frac{1}{3}MR^2 + 4MR^2 = \frac{13}{3}MR^2$$

Illustration Four holes of radius R are cut from a thin square plate of side 4R and mass M. Determine the moment of inertia of the remaining portion about Z–axis.



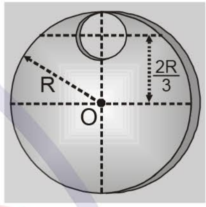
Solution. M = Mass of the square plate before the holes were cut.

Mass of each hole $m = \left[\frac{M}{16R^2}\right]\pi R^2 = \frac{\pi M}{16}$.

$\therefore$ Moment of inertia of the remaining portion,

$$I = I_{square} - 4I_{hole} = \frac{M}{12}(16R^2 + 16R^2) - 4\left[\frac{mR^2}{2} + m(\sqrt{2}R)^2\right] = \frac{8}{3}MR^2 - 10mR^2 = \left[\frac{8}{3} - \frac{10\pi}{16}\right]MR^2.$$

Illustration A thin uniform disc has a mass 9M and radius R. A disc of radius $\frac{R}{3}$ is cutoff as shown in figure. Find the moment of inertia of the remaining disc about an axis passing through O and perpendicular to the plane of disc.



Solution. As the mass is uniformly distributed on the disc,

so mass density (per unit area) $= \frac{9M}{\pi R^2}$. Mass of removed portion $= \frac{9M\pi}{\pi R^2} \times \left[\frac{R}{3}\right]^2 = M$

So the moment of inertia of the removed portion about the stated axis by theorem of parallel axes is :

$$I_1 = \frac{M}{2}\left[\frac{R}{3}\right]^2 + M\left[\frac{2R}{3}\right]^2 = \frac{MR^2}{2} \qquad .....(i)$$

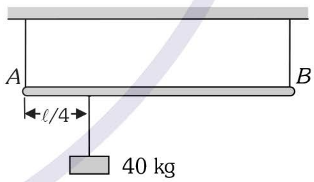
The moment of inertia of the original complete disc about the stated axis is I_2 then

$$I_2 = 9M\frac{R^2}{2} \qquad .....(ii)$$

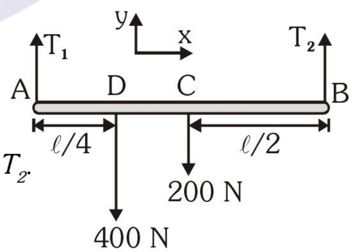
So the moment of inertia of the left over disc shown in fig. is $I_2 - I_1$.

i.e., $I_2 - I_1 = 4MR^2$.

Illustration A uniform rod of 20 kg is hanging in a horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tensions developed in each thread.



Solution. Free body diagram of the rod is shown in the figure.



Translational equilibrium requires

$\Sigma F_y = 0 \Rightarrow T_1 + T_2 = 400 + 200 = 600$ N (i)

Rotational equilibrium: Applying the condition about A, we get T_2.

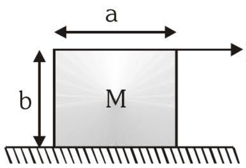
$\Sigma \vec{\tau}_A = \vec{0} \Rightarrow -400(l/4) - 200(l/2) + T_2 l = 0$

$T_2 = 200$ N

From equation (i)

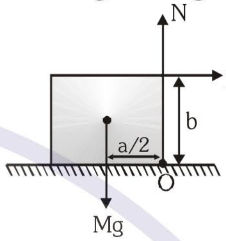
$T_1 = 400$ N.

Illustration Find the minimum value of F for the block to topple about an edge.



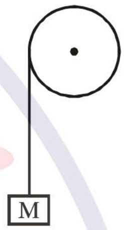
Solution. When the block is about to topple the normal reaction N shifts to the edge through O.

FBD during toppling Taking torque about O



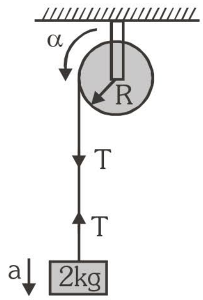
$$F(b) = Mg\left(\frac{a}{2}\right) \Rightarrow F_{min} = \frac{Mga}{2b}.$$

Illustration A fixed pulley of radius 20 cm and moment of inertia 0.32 kg·m^2 about its axle has a massless cord wrapped around its rim. A 2 kg mass M is attached to the end of the cord. The pulley can rotate about its axis without any friction. Find the acceleration of the mass M. (Assume g = 10 m/s^2)



Solution. For the motion of the block 2g – T=2a

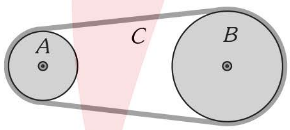
For the motion of the pulley $\tau = TR = I\alpha$



$$\because\ a = \alpha R \quad \therefore\ T = \frac{Ia}{R^2} \Rightarrow 2g - \frac{Ia}{R^2} = 2a \Rightarrow a = \frac{g}{1+\dfrac{I}{2R^2}}$$

$$\Rightarrow a = \frac{10}{1+\dfrac{0.32}{2\times 0.2\times 0.2}} = \frac{10}{1+4}\ \frac{10}{5} = 2\ ms^{-2}.$$

Illustration A belt moves over two pulleys *A* and *B* as shown in the figure. The pulleys are mounted on two fixed horizontal axles. Radii of the pulleys *A* and *B* are 50 cm and 80 cm respectively. Pulley *A* is driven at constant angular acceleration of 0.8 rad/s^2 until pulley *B* acquires an angular velocity of 10 rad/s. The belt does not slide on either of the pulleys.



(a) Find the acceleration of a point *C* on the belt and angular acceleration of the pulley *B*.
(b) How long does it take for the pulley *B* to acquire an angular velocity of 10 rad/s ?

Solution. Since the belt does not slide on the pulleys, magnitudes of velocity and acceleration of any point on the belt are same as that of any point on the periphery of either of the two pulleys.

$\vec{a}_T = \vec{\alpha} \times \vec{r}$ $\qquad a_C = \alpha_A r_A = \alpha_B r_B$

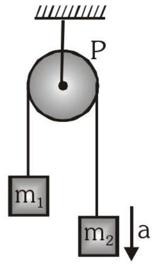
Substituting $r_A = 0.5$ m, $r_B = 0.8$ m and $\alpha_A = 0.8$ rad/s^2,

we have $a_C = 0.4$ m/s^2 and $\alpha_B = \dfrac{a_C}{r_B} = \dfrac{\alpha_A r_A}{r_B} = 0.5$ rad/s^2

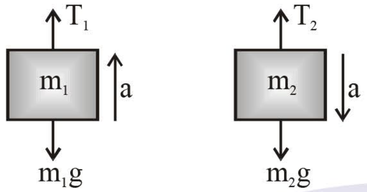
$$\because \quad \omega = \omega_o + \alpha t \Rightarrow t = \frac{\omega_B - \omega_{Bo}}{\alpha_B}$$

Substituting $\omega_{Bo} = 0$, $\omega_B = 10$ rad/s and $\alpha_B = 0.5$ rad/s^2, we have $t = 20$ s

Illustration In the figure, the blocks have unequal masses m_1 and m_2 ($m_1 < m_2$). m_2 has a downward acceleration a. The pulley P has a radius r, and some mass. The string does not slip on the pulley. Find the acceleration of block and angular acceleration of pulley. (I = momentum of inertia of pulley)



Solution.



$T_1 - m_1g = m_1a$ ....(1)

$m_2g - T_2 = m_2a$ ....(2)

Total torque = $I\alpha$

$(T_2 - T_1)R = I\alpha$ ....(3)

$a = \alpha R$ ....(4)

By Eq. (1), (2) (3) and (4) $\Rightarrow a = \dfrac{(m_2 - m_1)}{m_1 + m_2 + \dfrac{I}{r^2}} g \quad \Rightarrow \alpha = \dfrac{a}{R}$

Illustration A spherical shell has a radius of 1.90 m. An applied torque of 960 N-m gives the shell an angular acceleration of 6.20 rad/s^2 about an axis passing through the centre of the shell. What are (a) the rotational inertia of the shell about that axis and (b) the mass of the shell?

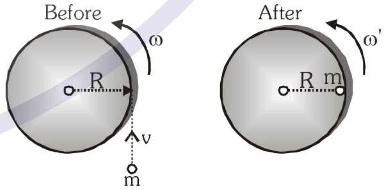
Solution. (a) Use $\tau = I\alpha$ where τ is the net torque acting on the shell, I is the rotational inertia of the shell, and α as its angular acceleration.

This gives $I = \dfrac{\tau}{\alpha} = \dfrac{960 \text{ N}\cdot\text{m}}{6.20 \text{ rad/s}^2} = 155 \text{ kg-m}^2$

(b) The rotational inertia of the shell is given by $I = \left(\dfrac{2}{3}\right)MR^2$

$\therefore M = \dfrac{3I}{2R^2} = \dfrac{3\left(155\text{kg}\cdot\text{m}^2\right)}{2(1.90 \text{ m})^2} = 64.4$ kg.

Illustration A solid cylinder of mass 'M' and radius 'R' is rotating along its axis with angular velocity ω without friction. A particle of mass 'm' moving with velocity v collides against the cylinder and sticks to its rim. After the impact calculate angular velocity of cylinder.



Solution. Initial angular momentum of cylinder = $I\omega$

Initial angular momentum of particle = mvR

Before collision the total angular momentum $L_1 = I\omega + mvR$

After collision the total angular momentum $L_2 = (I + mR^2)\omega'$

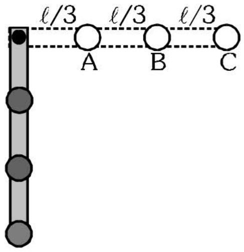
$L_1 = L_2 \Rightarrow (I+mR^2)\omega' = I\omega+mvR$.

New angular velocity $\omega' = \dfrac{I\omega + mvR}{I + mR^2}$.

Note : Initial kinetic energy of the system $= \dfrac{1}{2}I\omega^2 + \dfrac{1}{2}mv^2$.

Final kinetic energy of the system $= \dfrac{1}{2}(I+mR^2)\omega'^2$

Illustration A light rod carries three equal masses A, B and C as shown in the figure. What will be the velocity of B in the vertical position of the rod, if it is released from horizontal position as shown in the figure ?



(A) $\sqrt{\frac{8g\ell}{7}}$ (B) $\sqrt{\frac{4g\ell}{7}}$ (C) $\sqrt{\frac{2g\ell}{7}}$ (D) $\sqrt{\frac{10g\ell}{7}}$

Solution. Applying law of conservation of mechenical energy .
Loss in gravitational P.E. = Gain in rotational K.E. i.e.,

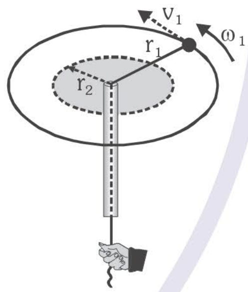
$$mg\frac{\ell}{3} + mg\left(\frac{2\ell}{3}\right) + mg\ell = \frac{1}{2}\left(m\left(\frac{\ell}{3}\right)^2 + m\left(\frac{2\ell}{3}\right)^2 + m\ell^2\right)\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{36g}{14\ell}} \Rightarrow v_B = \omega\ell_B = \frac{2\ell}{3}\sqrt{\frac{36g}{14\ell}} = \sqrt{\frac{8g\ell}{7}}.$$

Illustration A point mass is tied to one end of a cord whose other end passes through a vertical hollow tube, caught in one hand. The point mass is being rotated in a horizontal circle of radius 2 m with a speed of 4 m/s. The cord is then pulled down so that the radius of the circle reduces to 1m. Compute the new linear and angular velocities of the point mass and also the ratio of kinetic energies in the initial and final states.

Solution. The force on the point mass due to cord is radial and hence the torque about the centre of rotation is zero. Therefore, the angular momentum must remain constant as the cord is shortened.
Let mass of the particle be m let it rotate initially in circle of radius r_1 with linear velocity v_1 and angular velocity ω_1. Further let the corresponding quantities in the final state be radius r_2, linear velocity v_2 and angular velocity ω_2.
$\because$ Initial angular momentum = Final angular momentum

$$\therefore \quad I_1\omega_1 = I_2\omega_2 \Rightarrow mr_1^2\frac{v_1}{r_1} = mr_2^2\frac{v_2}{r_2} \Rightarrow r_1v_1 = r_2v_2$$



$$\therefore \quad v_2 = \frac{r_1}{r_2}v_1 = \frac{2}{1}\times 4 = 8 \text{ m/s} \quad \text{and} \quad \omega_2 = \frac{v_2}{r_2} = \frac{8}{1} = 8 \text{ rad/s}.$$

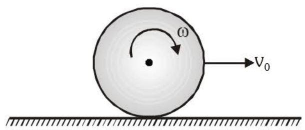
$$\frac{\text{Final K.E.}}{\text{Initial K.E.}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2} = \frac{mr_2^2\times\left[\frac{v_2}{r_2}\right]^2}{mr_1^2\times\left[\frac{v_1}{r_1}\right]^2} = \frac{v_2^2}{v_1^2} = \frac{(8)^2}{(4)^2} = 4.$$

Illustration A thin meter scale is kept vertical by placing its lower end hinged on floor. It is allowed to fall. Calculate the velocity of its upper end when it hits the floor .

Solution. Loss in PE $\left(\frac{mg\ell}{2}\right)$ = gain in rotational KE $\left(\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{m\ell^2}{3}\times\frac{v^2}{\ell^2}\right) \Rightarrow v = \sqrt{3g\ell}$.

Illustration A solid sphere rolls without slipping on a rough surface and the centre of mass has a constant speed v_0. If the mass of the sphere is m and its radius is R, then find the angular momentum of the sphere about the point of contact.

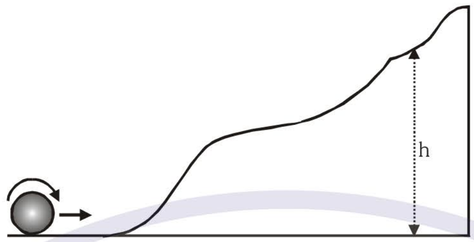
Solution $\because \vec{L}_P = \vec{L}_{cm} + \vec{r}\times\vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R}\times m\vec{v}_{cm}$; here $v_{cm} = v_0$



Since sphere is in pure rolling motion hence $\omega = v_0R$

$$\Rightarrow \vec{L}_P = \left(\frac{2}{5}MR^2\frac{v_0}{R}\right)\left(-\hat{k}\right) + Mv_0R\left(-\hat{k}\right) = \frac{7}{5}Mv_0R\left(-\hat{k}\right)$$

Illustration A body of mass M and radius r, rolling with velocity v on a smooth horizontal floor, rolls up a rough irregular inclined plane up to a vertical height ($3v^2/4g$). Compute the moment of inertia of the body and comment on its shape?



Solution The total kinetic energy of the body $E = E_t + E_r = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$\Rightarrow$ $E = \frac{1}{2}Mv^2\,[1 + (I/Mr^2)]$ [as $v = r\omega$]

When it rolls up on an irregular inclined plane of height $h = (3v^2/4g)$, its KE is fully converted into PE, so by conservation of mechanical energy $\frac{1}{2}Mv^2\left[1+\frac{I}{Mr^2}\right] = Mg\left[\frac{3v^2}{4g}\right]$ which on simplification gives $I = (1/2)\,Mr^2$. This result clearly indicates that the body is either a disc or a cylinder.