COLLISION AND CENTRE OF MASS

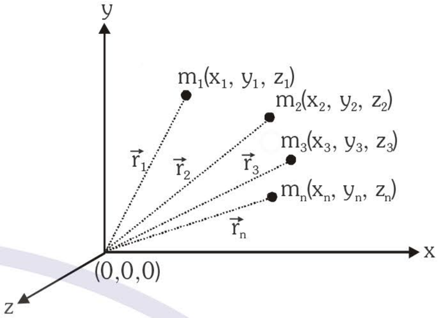
- **Centre of mass :** For a system of particles centre of mass is that point at which its total mass is supposed to be concentrated.
- **Centre of mass of system of discrete particles**

  Total mass of the body : $M = m_1 + m_2 + \ldots.. + m_n$ then

$$\vec{R}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{1}{M}\Sigma m_i\vec{r}_i$$

co-ordinates of centre of mass :

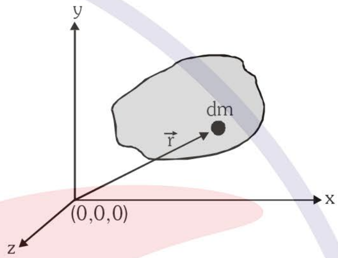
$$x_{cm} = \frac{1}{M}\Sigma m_i x_i \ , \ y_{cm} = \frac{1}{M}\Sigma m_i y_i \ \text{ and } \ z_{cm} = \frac{1}{M}\Sigma m_i z_i$$



- **Centre of mass of continuous bodies**

$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}dm \Rightarrow \vec{x}_{cm} = \frac{1}{M}\int \vec{x}\ dm\ ,\ \vec{y}_{cm} = \frac{1}{M}\int \vec{y}\ dm \ \text{ and } \ \vec{z}_{cm} = \frac{1}{M}\int \vec{z}\ dm$$

x, y, z are the co-ordinate of the COM of the dm mass.



The centre of mass after removal of a part of a body

Original mass (M) – mass of the removed part (m)

= {original mass (M)} + { – mass of the removed part (m)}

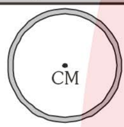
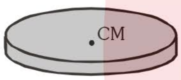
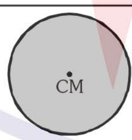
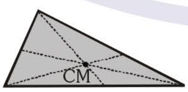
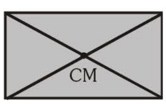
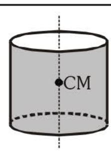
The formula changes to : $x_{CM} = \frac{Mx - mx'}{M - m}$; $y_{CM} = \frac{My - my'}{M - m}$; $y_{CM} = \frac{Mz - mz'}{M - m}$

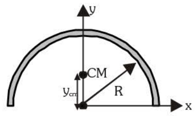
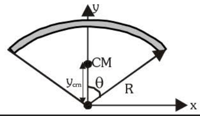
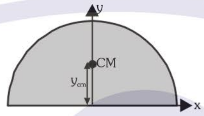
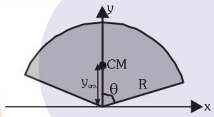
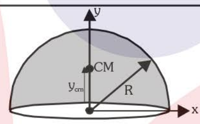
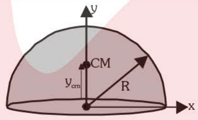
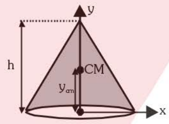
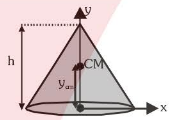
Where x', y' and z' represent the coordinates of the centre of mass of the removed part

Imporant points about centre of mass :

1. There may or may not be any mass present physically at the centre of mass.
2. Centre of mass may be inside or outside a body.
3. Centre of mass depends on the distribution of mass with in the body and is closer to massive portion.
4. For symmetrical bodies having homogeneous distribution of mass, centre of mass lie on the line of symmetry and can be on centre of symmetry or geometrical centre.

CENTRE OF MASS OF SOME COMMON OBJECTS

Body	Shape of body	Position of centre of mass
Uniform Ring		Centre of ring
Uniform Disc		Centre of disc
Uniform Rod		Centre of rod
Solid sphere/ hollow sphere		Centre of sphere
Triangular plane lamina		Point of intersection of the medians of the triangle i.e. centroid
Plane lamina in the form of a square or rectangle or parallelogram		Point of intersection of diagonals
Hollow/solid cylinder		Middle point of the axis of cylinder

Body	Shape of body	Position of centre of mass
Half ring		$y_{cm} = \frac{2R}{\pi}$
Segement of a ring		$y_{cm} = \frac{R\sin\theta}{\theta}$
Half disc (plate)		$y_{cm} = \frac{4R}{3\pi}$
Sector of a disc (plate)		$y_{cm} = \frac{2R\sin\theta}{3\theta}$
Hollow hemisphere		$y_{cm} = \frac{R}{2}$
Solid hemisphere		$y_{cm} = \frac{3R}{8}$
Hollow cone		$y_{cm} = \frac{h}{3}$
Solid cone		$y_{cm} = \frac{h}{4}$

MOTION OF CENTRE OF MASS

For a system of particles, velocity of centre of mass $\vec{v}_{CM} = \frac{d\vec{R}_{CM}}{dt} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots}{m_1 + m_2 + \ldots.}$

Similarly acceleration $\vec{a}_{CM} = \frac{d}{dt}\left(\vec{v}_{CM}\right) = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots}{m_1 + m_2 + \ldots.}$

- **Law of conservation of linear momentum**

  Linear momentum of a system of particles is equal to the product of mass of the system with velocity of its centre of mass.

  From Newton's second law $\vec{F}_{ext.} = \frac{d\left(M\vec{v}_{CM}\right)}{dt}$

  If $\vec{F}_{ext.} = \vec{0}$ then $M\vec{v}_{CM} = \text{constant}$ or $\vec{p}_{final} = \vec{p}_{initial}$

  If no external force acts on a system the velocity of its centre of mass remains constant, i.e., velocity of centre of mass is unaffected by internal forces. For example : Explosion of bomb at rest.

- **Impulse – Momentum theorem**

  Impulse of a force is equal to the change of momentum

  force time graph area gives change in momentum.

  $$\boxed{\int_{t_1}^{t_2} \vec{F}dt = \Delta\vec{p}}$$

Collision of bodies

The event or the process, in which two bodies either coming in contact with each other or due to mutual interaction at distance apart, affect each others motion (velocity, momentum, energy or direction of motion) is defined as a collision.

In collision

- The particles come closer before collision and after collision they either stick together or move away from each other.
- The particles need not come in contact with each other for a collision always. (But we genrally consiver those collision which are in contact).
- The law of conservation of linear momentum is necessarily applicable in a collision, whereas the law of conser-vation of mechanical energy is not.

TYPES OF COLLISION

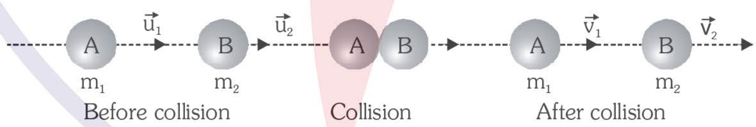
- On the basis of direction
  - one-dimensional collision or Head on collision → The collision, in which the particles move along the same straight line before and after the collision, is defined as one dimensional collision.
  - Two dimensional collision or Oblique collision → The collision, in which the particles move along the same plane at different angles before and after collision, is defined as oblique collision.
- On the basis of kinetic energy
  - Elastic collision → A collision is said to be elastic, if the total kinetic energy before and after collision remains the same
  - In-elastic collision → A collision is said to be inelastic, if the total kinetic energy does not remains constant
  - Perfectly In-elastic collision → The collision, in which particles gets sticked together after the collision, is called perfectly inelastic collision. In this type of inelastic collision, loss of energy is maximum.

- **Coefficient of restitution (Newton's law)**

$$e = -\frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}} = \left|\frac{\vec{v}_2 - \vec{v}_1}{\vec{u}_1 - \vec{u}_2}\right|$$

Value of e is 1 for elastic collision, 0 for perfectly inelastic collision and $0 < e < 1$ for inelastic collision.

- **Head on collision**



Head on inelastic collision of two particles

Let the coefficient of restitution for collision is e

(i) Momentum is conserved $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ ...(i)

(ii) Kinetic energy is not conserved.

(iii) According to Newton's law $e = \frac{v_2 - v_1}{u_1 - u_2}$ ...(ii)

By solving eq. (i) and (ii) :

$$v_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)u_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)u_2 = \frac{m_1u_1 + m_2u_2 - m_2e(u_1 - u_2)}{m_1 + m_2}$$

$$v_2 = \left(\frac{m_2 - em_1}{m_1 + m_2}\right)u_2 + \left(\frac{(1+e)m_1}{m_1 + m_2}\right)u_1 = \frac{m_1u_1 + m_2u_2 - m_1e(u_2 - u_1)}{m_1 + m_2}$$

Elastic Collision (e=1)

- **If the two bodies are of equal masses :** $m_1 = m_2 = m$, $v_1 = u_2$ and $v_2 = u_1$

  Thus, if two bodies of equal masses undergo elastic collision in one dimension, then after the collision, the bodies will exchange their velocities.

- **If the mass of a body is negligible as compared to other.**

  If $m_1 >> m_2$ and $u_2 = 0$ then $v_1 = u_1$, $v_2 = 2u_1$

  When a heavy body A collides against a light body B at rest, the body A should keep on moving with same velocity and the body B will move with velocity double that of A.

  If $m_2 >> m_1$ and $u_2 = 0$ then $v_2 = 0$, $v_1 = -u_1$

  When light body A collides against a heavy body B at rest, the body A should start moving with same velocity just in opposite direction while the body B should practically remains at rest.

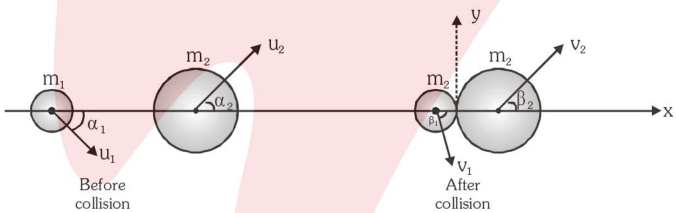
- **Loss in kinetic energy in inelastic collision**

$$\Delta K = \frac{m_1 m_2}{2(m_1 + m_2)}(1 - e^2)|\vec{u}_1 - \vec{u}_2|^2$$

Oblique Collision

Conserving the momentum of system in directions along normal (x axis in our case)

$$m_1u_1\cos\alpha_1 + m_2u_2\cos\alpha_2 = m_1v_1\cos\beta_1 + m_2v_2\cos\beta_2$$



Since no force is acting on m_1 and m_2 along the tangent (i.e. y–axis) the individual momentum of m_1 and m_2 remains conserved.

$$m_1u_1\sin\alpha_1 = m_1v_1\sin\beta_1 \ \& \ m_2u_2\sin\alpha_2 = m_2v_2\sin\beta_2$$

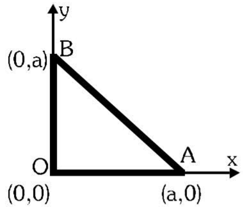
By using Newton's experimental law along the line of impact

$$e = \frac{v_2\cos\beta_2 - v_1\cos\beta_1}{u_1\cos\alpha_1 - u_2\cos\alpha_2}$$

KEY POINTS

- Sum of mass moments about centre of mass is zero. i.e. $\sum m_i \vec{r}_{i/cm} = \vec{0}$
- A quick collision between two bodies is more violent then slow collision, even when initial and final velocities are equal because the rate of change of momentum determines that the impulsive force small or large.
- Heavy water is used as moderator in nuclear reactors as energy transfer is maximum if $m_1 \simeq m_2$.
- Impulse momentum theorem is equivalent to Newton's second law of motion.
- For a system, conservation of linear momentum is equivalent to Newton's third law of motion.

Illustration Three rods of the same mass are placed as shown in the figure. Calculate the coordinates of the centre of mass of the system.



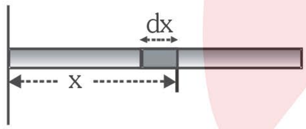
Solution CM of rod OA is at $\left(\frac{a}{2},0\right)$, CM of rod OB is at $\left(0,\frac{a}{2}\right)$ and CM of rod AB is at $\left(\frac{a}{2},\frac{a}{2}\right)$

For the system, $x_{cm} = \frac{m\times\frac{a}{2}+m\times 0+m\times\frac{a}{2}}{m+m+m} = \frac{a}{3} \Rightarrow y_{cm} = \frac{m\times 0+m\times\frac{a}{2}+m\times\frac{a}{2}}{m+m+m} = \frac{a}{3}$

Illustration If the linear density of a rod of length L varies as $\lambda = A + Bx$, determine the position of its centre of mass. (where x is the distance from one of its ends)

Solution Let the X–axis be along the length of the rod with origin at one of its end as shown in figure. As the rod is along x–axis, so, $y_{CM} = 0$ and $z_{CM} = 0$ i.e., centre of mass will be on the rod. Now consider an element of rod of length dx at a distance x from the origin, mass of this element $dm = \lambda dx = (A + Bx)dx$ so,

$$x_{CM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L x(A+Bx)dx}{\int_0^L (A+Bx)dx} = \frac{\frac{AL^2}{2}+\frac{BL^3}{3}}{AL+\frac{BL^2}{2}} = \frac{L(3A+2BL)}{3(2A+BL)}$$



Note :

(i) If the rod is of uniform density then $\lambda = A$ = constant & $B = 0$ then $x_{CM} = L/2$

(ii) If the density of rod varies linearly with x, then $\lambda = Bx$ and $A = 0$ then $x_{CM} = 2L/3$

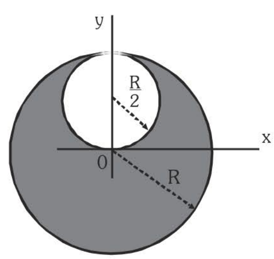
Illustration A disc of radius R is cut off from a uniform thin sheet of metal. A circular hole of radius $\frac{R}{2}$ is now cut out from the disc, with the hole being tangent to the rim of the disc. Find the distance of the centre of mass from the centre of the original disc.

Solution We treat the hole as a 'negative mass' object that is combined with the original uncut disc. (When the two are overlapped together, the hole region then has zero mass). By symmetry, the CM lies along the +y–axis in figure, so $x_{CM} = 0$. With the origin at the centre of the original circle whose mass is assumed to be m.

Mass of original uncut circle $m_1 = m$ & Location of CM = (0,0)

Mass of hole of negative mass : $m_2 = \frac{m}{4}$; Location of CM = $\left(0,\frac{R}{2}\right)$

Thus $y_{CM} = \frac{m_1y_1+m_2y_2}{m_1+m_2} = \frac{m(0)+\left(-\frac{m}{4}\right)\frac{R}{2}}{m+\left(-\frac{m}{4}\right)} = -\frac{R}{6}$

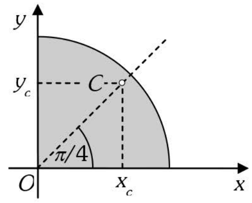


So the centre of mass is at the point $\left(0,-\frac{R}{6}\right)$.

Thus, the required distance is R/6.

Illustration Find coordinates of center of mass of a quarter sector of a uniform disk of radius r placed in the first quadrant of a Cartesian coordinate system with centre at origin.

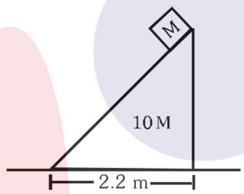
Solution. From the result obtained for sector of circular plate distance OC of the center of mass form the center is



$$OC = \frac{2r\sin(\pi/4)}{3\pi/4} = \frac{4\sqrt{2}r}{3\pi}$$

Coordinates of the center of mass (x_c, y_c) are $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$

Illustration A block of mass M is placed on the top of a bigger block of mass 10M as shown in figure. All the surfaces are frictionless. The system is released from rest.



Find the distance moved by the bigger block at the instant when the smaller block reaches the ground.

Solution If the bigger block moves toward right by a distance (x) then the smaller block will move toward left by a distance (2.2 – x).

Now considering both the blocks together as a system, horizontal position of CM remains same.

As the sum of mass moments about centre of mass is zero i.e. $\sum m_i x_{i/cm} = 0$.

$M(2.2 - x) = 10\ Mx \Rightarrow x = 0.2$ m.

Illustration A man of mass 80 kg stands on a plank of mass 40 kg. The plank is lying on a smooth horizontal floor. Initially both are at rest. The man starts walking on the plank towards north and stops after moving a distance of 6 m on the plank. Then

(A) the centre of mass of plank-man system remains stationary.

(B) the plank will slide to the north by a distance of 4 m

(C) the plank will slide to the south by a distance of 4 m

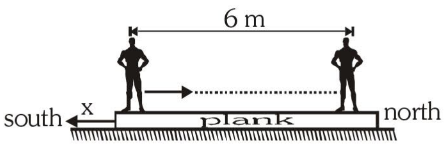
(D) the plank will slide to the south by a distance of 12 m

Solution **Ans. (A,C)**

Since net force is zero so centre of mass remains stationary

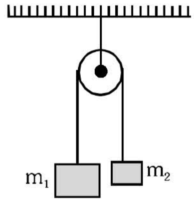
Let x be the displacement of the plank.

Since CM of the system remains stationary



so $80\ (6-x) = 40\ x \Rightarrow 12 - 2x = x \Rightarrow x = 4$ m.

Illustration Two bodies of masses m_1 and m_2 ($<m_1$) are connected to the ends of a massless cord and allowed to move as shown in figure. The pulley is massless and frictionless. Calculate the acceleration of the centre of mass.



Solution If $\vec{a}$ is the acceleration of m_1, then $-\vec{a}$ is the acceleration of m_2 then

$$\vec{a}_{cm} = \frac{m_1\vec{a} + m_2(-\vec{a})}{m_1 + m_2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{a}$$

But $\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{g}$ so $\vec{a}_{cm} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \vec{g}$.

Illustration In a gravity free room a man of mass m_1 is standing at a height h above the floor. He throws a ball of mass m_2 vertically downward with a speed u. Find the distance of the man from the floor when the ball reaches the ground.

Solution Time taken by ball to reach the ground $t = \frac{h}{u}$

By conservation of linear momentum, speed of man $v = \left(\frac{m_2 u}{m_1}\right)$

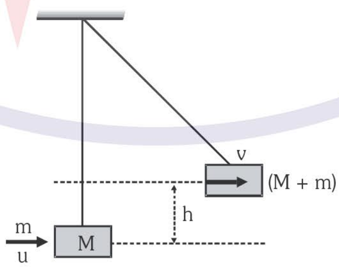
Therefore, the man will move upward by a distance $= vt = \left(\frac{h}{u}\right)\left(\frac{m_2 u}{m_1}\right) = \frac{m_2}{m_1}h$

Total distance of the man from the floor $= h + \frac{m_2}{m_1}h = \left(1 + \frac{m_2}{m_1}\right)h$.

Illustration A simple pendulum of length 1m has a wooden bob of mass 1kg. It is struck by a bullet of mass 10^{-2} kg moving with a speed of 2×10^2 m/s. The bullet gets embedded within the bob. Obtain the height to which the bob rises before swinging back.

Solution Applying principle of conservation of linear momentum

$$mu = (M + m)\,v \Rightarrow 10^{-2} \times (2 \times 10^2) = (1 + .01)\,v \Rightarrow v = \frac{2}{1.01} \text{ m/s}$$



Initial KE of the block with bullet in it, is fully converted into PE as it rises through a height h, given by

$$\frac{1}{2}(M+m)v^2 = (M+m)gh \Rightarrow v^2 = 2gh \Rightarrow h = \frac{v^2}{2g} = \left(\frac{2}{1.01}\right)^2 \times \frac{1}{2 \times 9.8} = 0.2 \text{ m.}$$

Illustration A body falling on the ground from a height of 10 m, rebounds to a height 2.5 m calculate the: (i) percentage loss in K.E. (ii) ratio of the velocities of the body just before and after the collision.

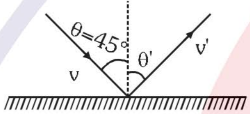
Solution Let v_1 and v_2 be the velocities of the body just before and just after the collision.

$$KE_1 = \frac{1}{2}mv_1^2 = mgh_1 \ldots(i) \text{ and } KE_2 = \frac{1}{2}mv_2^2 = mgh_2 \ldots(ii) \Rightarrow \frac{v_1^2}{v_2^2} = \frac{h_1}{h_2} = \frac{10}{2.5} = 4 \Rightarrow \frac{v_1}{v_2} = 2.$$

$$\text{Percentage loss in KE} = \frac{mg(h_1 - h_2)}{mgh_1} \times 100 = \frac{10 - 2.5}{10} \times 100 = 75\%.$$

Illustration A ball of mass m hits a floor with a speed v making an angle of incidence $\theta = 45°$ with the normal to the floor. If the coefficient of restitution is $e = \frac{1}{\sqrt{2}}$, find the speed of the reflected ball and the angle of reflection. **[AIPMT (Mains) 2005]**

Solution



Since the floor exerts a force on the ball along the normal during the collision so horizontal component of velocity remains same and only the vertical component changes.

Therefore, $v'\sin\theta' = v\sin\theta = \frac{v}{\sqrt{2}}$

and $v'\cos\theta' = ev\cos\theta = \frac{1}{\sqrt{2}} v \times \frac{1}{\sqrt{2}} = \frac{v}{2}$.

$$\Rightarrow \quad v'^2 = \frac{v^2}{2} + \frac{v^2}{4} = \frac{3}{4} v^2 \Rightarrow v' = \frac{\sqrt{3}}{2} v$$

and $\tan\theta' = \sqrt{2} \Rightarrow \theta' = \tan^{-1}\sqrt{2}$.

Illustration A ball of mass 2 kg moving with a speed of 5 m/s collides directly with another ball of mass 3 kg moving in the same direction with a speed of 4 m/s. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution Denoting the first ball by *A* and the second ball by *B*, velocities immediately before and after the impact are shown in the figure.

By COLM : $2(5) + 3(4) = 2v_1 + 3v_2 \Rightarrow 2v_1 + 3v_2 = 22$ ...... (i)

By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow \frac{2}{3} = \frac{v_2 - v_1}{5 - 4} \Rightarrow 3v_2 - 3v_1 = 2$ ...... (ii)

by solving equations (i) and (ii), we have $v_1 = 4$ m/s and $v_2 = 4.67$ m/s