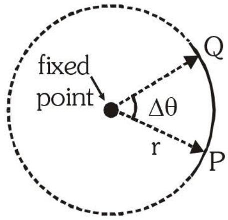
CIRCULAR MOTION

When a particle moves in a plane such that its distance from a fixed point remains constant then its motion is called as circular motion with respect to that fixed point. That fixed point is called centre and the distance is called radius of circular path.

Kinematics of Circular Motion

Angular Displacement

Angle traced by position vector $\Delta\theta$ = angular displacement



$$\text{Angle} = \frac{\text{Arc}}{\text{Radius}} \quad \Rightarrow \quad \Delta\theta = \frac{\text{Arc (PQ)}}{r}$$

Frequency (n)

Number of revolutions described by particle per second is its frequency. Its unit is revolutions per second (r.p.s.) or revolutions per minute (r.p.m.) r.p.m. = 60 r.p.m.

Time Period (T)

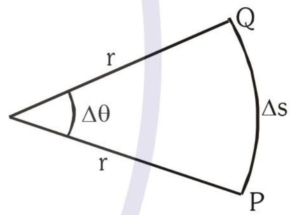
It is time taken by particle to complete one revolution. $T = \frac{1}{n}$

Angular Velocity (ω)

It is defined as the rate of change of angular position of moving particle .

$$\omega = \frac{\text{Angle traced}}{\text{Time taken}} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Relation between linear and Angular velocity



$$\text{Angle} = \frac{\text{Arc}}{\text{Radius}} \quad \text{or} \quad \Delta\theta = \frac{\Delta s}{r} \quad \text{or} \quad \Delta s = r\Delta\theta$$

$$\therefore \quad \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} \quad \text{if } \Delta t \to 0 \text{ then} \quad \frac{ds}{dt} = r\frac{d\theta}{dt} \qquad v = \omega r$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$ (direction of $\vec{v}$ is according to right hand thumb rule)

Average Angular Velocity (ω_{avg})

$$\omega_{avg} = \frac{\text{total angle of rotation}}{\text{total time taken}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} = 2\pi n$$

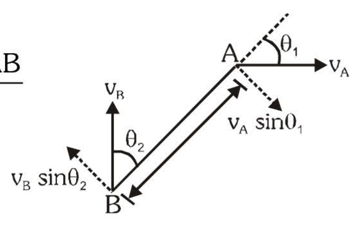
where θ_1 and θ_2 are angular position of the particle at instant t_1 and t_2.

Instantaneous Angular Velocity ($W_{inst.}$)

The angular velocity at a particular instant $(\omega) = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$ or $\vec{\omega} = \frac{d\vec{\theta}}{dt}$

Relative Angular Velocity

$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{\text{Relative velocity of A w.r.t. B perpendicular to line AB}}{\text{seperation between A and B}}$$



here $(v_{AB})_\perp = v_A \sin\theta_1 + v_B \sin\theta_2 \qquad \therefore \omega_{AB} = \frac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$

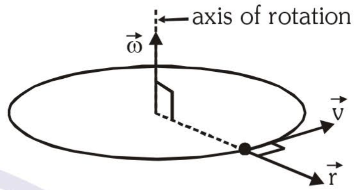
Angular Acceleration (α)

Rate of change of angular velocity is called angular acceleration.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \quad \text{or} \quad \vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Its an axial vector quantity. It direction is along the axis of rotation according to if $\omega\uparrow \Rightarrow \vec{\omega} \parallel \vec{\alpha}$, if $\omega\downarrow \Rightarrow \vec{\omega} \upharpoonleft\downharpoonright \vec{\alpha}$

Unit $\rightarrow$ rad/s^2



Relation between Angular and Linear Acceleration $\vec{v} = \vec{\omega} \times \vec{r}$

or $\quad \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \qquad \left(\frac{d\vec{\omega}}{dt} = \vec{\alpha} \text{ and } \frac{d\vec{r}}{dt} = \vec{v}\right)$

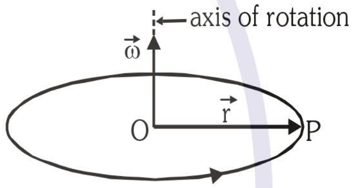
or $\quad \vec{a} = \vec{a}_T + \vec{a}_C \quad (\vec{a}_T = \vec{\alpha} \times \vec{r}$ is tangential acc. and $\vec{a}_C = \vec{\omega} \times \vec{v}$ is centripetal acceleration.)

$\therefore \quad |\vec{a}| = \sqrt{a_T^2 + a_C^2} \quad (\vec{a}_T$ and $\vec{a}_C$ are two component of net linear acceleration.)

Tangential Acceleration

$\vec{a}_T = \vec{\alpha} \times \vec{r}$, its direction is parallel to velocity.

$\vec{v} = \vec{\omega} \times \vec{r} \quad$ and $\quad \vec{a}_T = \vec{\alpha} \times \vec{r}$



Centripetal acceleration

$\vec{a}_C = \vec{\omega} \times \vec{v} \Rightarrow \vec{a}_C = \vec{\omega} \times (\vec{\omega} \times \vec{r}) \qquad (\because \vec{v} = \vec{\omega} \times \vec{r})$

Magnitude of centripetal acceleration, $a_C = \omega v = \frac{v^2}{r} = \omega^2 r$; $\qquad \vec{a}_C = \frac{v^2}{r}(-\hat{r})$

Equations of Motion in Circular Kinematics

If a particle moves along a circle with constant angular acceleration.

(i) $\omega = \omega_0 + \alpha t$

(ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 = \omega_0^2 + 2\alpha\theta$

(iv) $\theta = \left(\frac{\omega_0 + \omega}{2}\right)t$

(v) $\theta_n = \omega_0 + \frac{\alpha}{2}(2n - 1)$

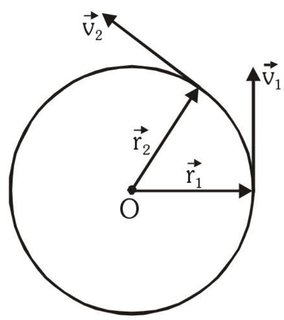
where, ω_0 = Initial angular velocity,

ω = Instantaneous angular velocity $\qquad \alpha$ = Angular acceleration,

θ = Angular displacement at time t, $\qquad \theta_n$ = Angular displacement in nth second

Uniform Circular Motion

When a particle moves in a circle at a constant speed then the motion is said to be a uniform circular motion.



Speed is constant, so that $\vec{a}_T = 0$

Acceleration of particle $\vec{a} = \vec{a}_C = \vec{\omega} \times \vec{v}$ or $a = \omega v \qquad \left(\text{but } \omega = \frac{v}{r}\right)$

$$\therefore \quad a = \frac{v^2}{r} = \omega^2 r = \text{centripetal acceleration}$$

due to centripetal acceleration, the velocity of the particle keeps on changing the direction i.e. the particle is accelerated.

Non- Uniform Circular Motion

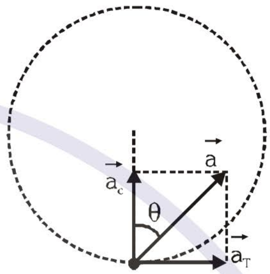
When a particle moving in a circle and if the speed of particle increases or decreses then the motion is non-uniform circular motion.

In non- uniform circular motion $|\vec{v}| \neq$ constant $\qquad \omega \neq$ constant

In non-uniform circular motion particle has two acceleration :

(a) Tangential acceleration $a_T = \dfrac{dv}{dt}$ = rate of change of speed; $v = \dfrac{ds}{dt}$ = speed; s=arc-length.

(b) Centripetal acceleration $a_c = \dfrac{v^2}{r} = \omega^2 r$

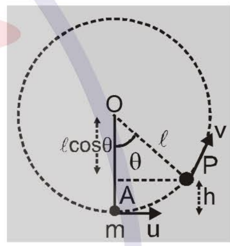


Net acceleration of the particle $\vec{a} = \vec{a}_c + \vec{a}_T \Rightarrow a = \sqrt{a_C^2 + a_T^2}$

$\therefore \ \theta = \tan^{-1}\left(\dfrac{a_T}{a_C}\right)$

CIRCULAR MOTION IN VERTICAL PLANE

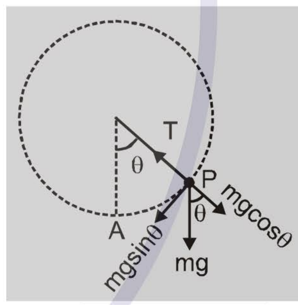
By conservation of mechanical energy between A and P.



$$0 + \frac{1}{2}mu^2 = mgh + \frac{1}{2}mv^2 \qquad \text{or} \qquad v = \sqrt{u^2 - 2gh}$$

or $\quad v = \sqrt{u^2 - 2g\ell(1 - \cos\theta)} \qquad [\because \ h = \ell(1 - \cos\theta)]$

Tension in string

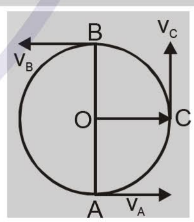


At point 'p' required centripetal force $= \dfrac{mv^2}{\ell}$

Net force toward centre

$$T - mg\cos\theta = \frac{mv^2}{\ell}$$

$$T = m\left[g\cos\theta + \frac{v^2}{\ell}\right] = \frac{m}{\ell}\left[u^2 - g\ell(2 - 3\cos\theta)\right]$$



$$T_A = \frac{mv_A^2}{\ell} + mg = \frac{mu^2}{\ell} + mg \quad [\theta = 0°]$$

$$T_B = \frac{mv_B^2}{\ell} - mg = \frac{mu^2}{\ell} - 5mg$$ $(\theta = 180°)$ [by law of conservation of energy $v_B^2 = u^2 - 2g(2\ell)$]

When particle is at point C of the circle –

$$T_C = \frac{mv_C^2}{\ell} = \frac{mu^2}{\ell} - 2mg$$ $(\theta = 90°)$ [by law of conservation of energy $v_C^2 = u^2 - 2g(\ell)$]

Thus we can conclude

$T_A > T_C > T_B$

$T_A - T_B = 6mg$

$T_A - T_C = 3mg$

$T_C - T_B = 3mg$

Cases

(i) if $u > \sqrt{5g\ell}$

In this case tension in the string will not be zero at any point, which implies that the particle will continue the circular motion.

(ii) $u = \sqrt{5g\ell} \rightarrow$ just completes the loop (at point B tension becomes zero, $T_B = \frac{mu^2}{\ell} - 5mg$)

critical velocity to complete the loop is $u = v_A = \sqrt{5g\ell}$

$v_B = \sqrt{g\ell}$ $[\, v_B^2 = v_A^2 - 4g(\ell)\,]$

$v_c = \sqrt{3g\ell}$ $[\, v_C^2 = v_A^2 - 2g(\ell)]$ and $T_A = 6mg$, $T_B = 0$, $T_c = 3mg$

(iii) $\sqrt{2g\ell} < u < \sqrt{5g\ell} \Rightarrow$ not complete the loop and leave the circular path.

Tension becomes zero between points C and B but speed $v \neq 0$ in this case. It moves in parabolic path beyond that point where tension become zero upto the point at same height in other half cicular part.

(iv) $u = \sqrt{2g\ell} \Rightarrow T = 0$ and $v = 0$ at point C.

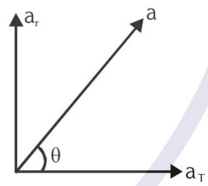
Particle will oscillate about point A.

(v) $u < \sqrt{2g\ell}$ $v = 0$ in between A and C, but $T \neq 0$ and particle will oscillate about 'A'.

Illustration A road makes a 90° bend with a radius of 190 m. A car enters the bend moving at 20 m/s. Finding this too fast, the driver decelerates at 0.92 m/s². Determine the acceleration of the car when its speed rounding the bend has dropped to 15 m/s.

Solution Since it is rounding a curve, the car has a radial acceleration associated with its changing direction, in addition to the tangential deceleration that changes its speed. We are given that $a_T = -0.92$ m/s², since the car is slowing down, the tangential acceleration is directed opposite the velocity.

The radial acceleration is $a_r = \frac{v^2}{r} = \frac{(15)^2}{190} = 1.2 \quad m/s^2$

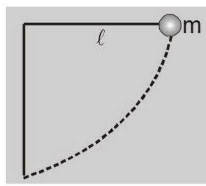


Magnitude of net acceleration. $a = \sqrt{a_r^2 + a_T^2} = [(1.2)^2 + (0.92)^2]^{1/2} = 1.5$ m/s²

and points at an angle $\theta = \tan^{-1}\left(\frac{a_r}{a_T}\right) = \tan^{-1}\left(\frac{1.2}{0.92}\right) = 53°$

relative to the tangent line to the circle.

Illustration A particle of mass 'm' tied with a string of length ℓ is released from horizontal position as shown is fig. Find the velocity at the lowest position.

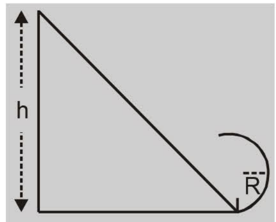


Solution apply COME $mg\ell = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2g\ell}$

Illustration A 4 kg ball swing in a vertical circle at the end of a cord 1m long. Find the maximum speed at which it can swing if the cord can sustain maximum tension of 104 N. (g = 10 m/s²)

Solution $T = \frac{mv^2}{r} + mg \therefore v = 4$ m/s

Illustration A ball is released from height 'h' as shown, which of the condition hold good for the particle to complete the circular path.

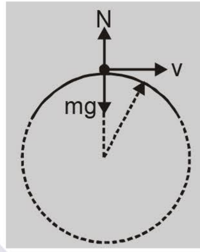


Solution $v = \sqrt{2gh} \geq \sqrt{5gR}$

$\Rightarrow \sqrt{2gh} \geq \sqrt{5gR} \Rightarrow h \geq \frac{5}{2}R$

Illustration A circular overbridge having radius 20m, what is the maximum speed with which a car can cross the bridge without leaving contact with the overbridge at the highest point ($g = 9.8\ m/sec^2$)

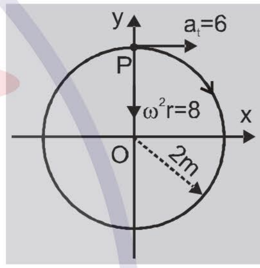
Solution For motion, $mg - N = \frac{mv^2}{r}$



When reaction N becomes zero, contact is about to leave

$\therefore\ mg = \frac{mv^2}{r}$ or $v = \sqrt{rg} = 14\ m/s$

Illustration A ring rotates about z-axis as shown in fig. in x-y plane. At a certain instant the acceleration of a particle P (Shown in fig.) on the ring is $(6\hat{i} - 8\hat{j})\ m/sec^2$. At that instant what is the angular acceleration and angular velocity of the ring ?



Solution $\vec{a} = \vec{a}_T + \vec{a}_C$ here $\vec{a}_C$ is along $-\hat{j}$ and $\vec{a}_T$ is along $\hat{i}$

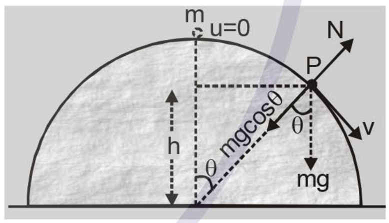
given $\vec{a} = 6\hat{i} - 8\hat{j} \Rightarrow a_T = 6 = \alpha r$

and $a_C = 8 = \omega^2 r$

now $\vec{\alpha} = \frac{6}{2} = 3\ rad/s^2(-\hat{k})$, $\vec{\omega} = 2\ rad/s(-\hat{k})$

Illustration A particle of mass 'm' slide down form the vertex of hemisphere, without any initial velocity. At what height from the horizontal will the particle leave the sphere.

Solution At point P $mg\cos\theta - N = \frac{mv^2}{R}$



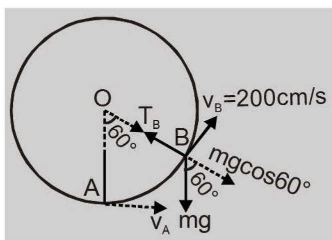
To leave the contact $N = 0$ so $v^2 = gR\cos\theta$

by law of conservation of energy $0 + mg\,R = \frac{1}{2}mv^2 + mgh$

$v^2 = 2g(R - h) = gR\cos\theta$ $(\because \cos\theta = \frac{h}{R})$ $\Rightarrow h = \frac{2}{3}R$

Illustration A particle of mass 100 gm is suspended from the end of a weightless string of length 100cm and is allowed to swing in a vertical plane. The speed of the mass is 200 cm/s, when the string makes an angle of 60° with the vertical axis. Determine the tension in the string at 60°.

Solution Tension at point (B)

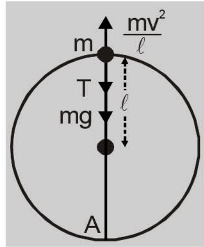


$F_{\text{net towards O}} = F_c \Rightarrow T_B - mg\cos 60° = \frac{mv_B^2}{\ell}$,

$T_B = \frac{mv_B^2}{\ell} + mg\cos 60° = \frac{100(200)^2}{100} + 100 \times (1000) \times \frac{1}{2}$

$= 40{,}000 + 50{,}000 = 90{,}000$ dyne $= 0.9 \times 10^5$ dyne $= 0.9$ N

Illustration In a vertical circular motion, tension at the highest point is equal to the weight of the particle, then find the speed and tension at the lowest point. Mass of the particle is m = 10 kg and length of string is ℓ = 10 m (g = 10 m/s²)



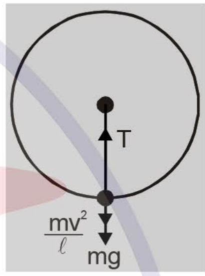
Solution At the highest point $T + mg = \frac{mv^2}{\ell}$ given T = mg

$$\therefore \quad mg + mg = \frac{mv^2}{\ell} \Rightarrow v^2 = 2\ell g \Rightarrow \quad v = \sqrt{2\ell g}$$

by conservation of mechanical energy between top most & lowest point

$$\frac{1}{2}mv^2 + mg\,(2\ell) = \frac{1}{2}\,mv_1^2$$

$$\Rightarrow \frac{1}{2}m\,(2\ell g) + 2mg\ell = \frac{1}{2}mv_1^2$$

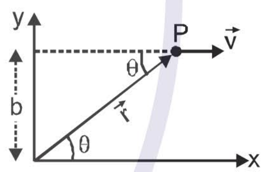


$$6g\ell = v_1^2 \quad \Rightarrow v_1 = \sqrt{6g\,\ell}$$

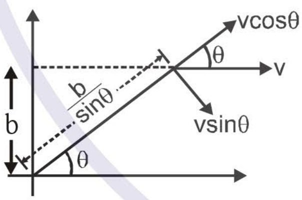
At lowest point $T = mg + \frac{mv_1^2}{\ell} = mg + \frac{m}{\ell} \times 6g\ell$

$T = 7mg = 700$ N and $v_1 = \sqrt{6g\,\ell} = 10\sqrt{6}$ m/s = 24.49 m/s

Illustration A particle is moving parallel to x–axis as shown in fig. such that the y component of its position vector is constant at all instants and is equal to 'b'. Find the angular velocity of the particle about the origin when its radius vector makes an angle θ with the x-axis.

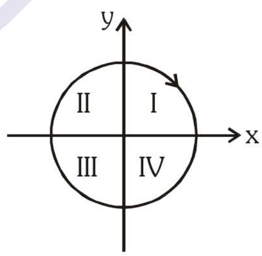


Solution



$$\therefore \quad \omega_{PO} = \frac{v\sin\theta}{\frac{b}{\sin\theta}} = \frac{v}{b}\sin^2\theta$$

Illustration A particle is moving in clockwise direction in a circular path as shown in figure. The instantaneous velocity of particle at a certain instant is $\vec{v} = (3\hat{i} + 3\hat{j})$ m/s. Then in which quardant does the particle lie at that instant? Explain your answer.



Solution II quardant. According to following figure x & y components of velocity are positive when the particle is in II quardant.

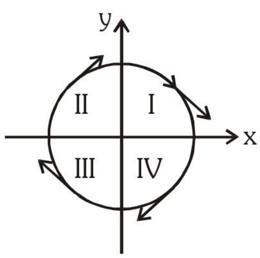
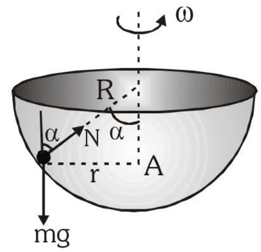


Illustration A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α . Find the angular speed at which the bowl is rotating.



Solution

$N \cos\alpha = mg$ .....(1)

$N \sin\alpha = mr\omega^2$ .....(2)

$r = R\sin\alpha$ .....(3)

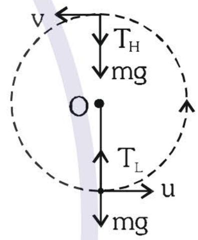
From equations (2) & (3)

$N\sin\alpha = m\omega^2 R\sin\alpha$

$N = mR\omega^2$ .....(4)

$\Rightarrow (mR\omega^2)\cos\alpha = mg \quad \Rightarrow \omega = \sqrt{\dfrac{g}{R\cos\alpha}}$

Illustration A particle of mass m tied to a string of length ℓ and given a circular motion in the vertical plane. If it performs the complete loop motion then prove that difference in tensions at the lowest and the highest point is 6 mg.



Solution Let the speeds at the lowest and highest points be u and v respectively.

At the lowest point, tension $= T_L = mg + \dfrac{mu^2}{\ell}$ ...(i)

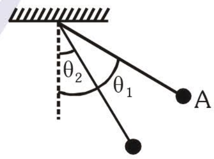
At the highest point, tension $= T_H = \dfrac{mv^2}{\ell} - mg$ . ..(ii)

By conservation of mechanical energy, $\dfrac{mu^2}{2} - \dfrac{mv^2}{2} = mg(2\ell) \quad \Rightarrow \quad u^2 = v^2 + 4g\ell$

Substituting this in eqn (i) $T_L = mg + \dfrac{m\left[v^2 + 4g\ell\right]}{\ell}$ ...(iii)

$\therefore$ From eq^n. (ii) & (iii) $T_L - T_H = 6mg$

Illustration A particle of mass m is connected to a light inextensible string of length ℓ such that it behaves as a simple pendulum. Now the string is pulled to point A making an angle θ_1 with the vertical and is released then obtain expressions for the :



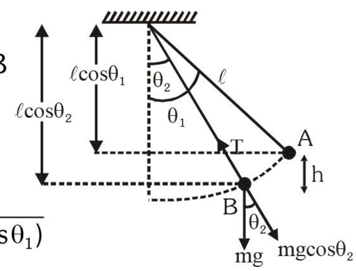
(i) speed of the particle and **[AIPMT (Mains) 2008]**

(ii) the tension in the string when it makes an angle θ_2 with the vertical.

Solution (i) $h = \ell(\cos\theta_2 - \cos\theta_1)$

Applying conservation of mechanical energy between points A & B

$\dfrac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh} = \sqrt{2g\ell(\cos\theta_2 - \cos\theta_1)}$



(ii) At position B, $T - mg\cos\theta_2 = \dfrac{mv^2}{\ell}$ where $v = \sqrt{2g\ell(\cos\theta_2 - \cos\theta_1)}$

$\Rightarrow T = mg\cos\theta_2 + \dfrac{m}{\ell}[2g\ell(\cos\theta_2 - \cos\theta_1)] = mg(3\cos\theta_2 - 2\cos\theta_1)$.