6.1 Introduction

Special cases often dominate our study of physics, and circular motion is certainly no exception. We see circular motion in many instances in the world; a bicycle rider on a circular track, a ball spun around by a string, and the rotation of a spinning wheel are just a few examples. Various planetary models described the motion of planets in circles before any understanding of gravitation. The motion of the moon around the earth is nearly circular. The motions of the planets around the sun are nearly circular. Our sun moves in nearly a circular orbit about the center of our galaxy, 50,000 light years from a massive black hole at the center of the galaxy.

We shall describe the kinematics of circular motion, the position, velocity, and acceleration, as a special case of two-dimensional motion. We will see that unlike linear motion, where velocity and acceleration are directed along the line of motion, in circular motion the direction of velocity is always tangent to the circle. This means that as the object moves in a circle, the direction of the velocity is always changing. When we examine this motion, we shall see that the direction of change of the velocity is towards the center of the circle. This means that there is a non-zero component of the acceleration directed radially inward, which is called the *centripetal acceleration*. If our object is increasing its speed or slowing down, there is also a non-zero *tangential acceleration* in the direction of motion. But when the object is moving at a constant speed in a circle then only the centripetal acceleration is non-zero.

In all of these instances, when an object is constrained to move in a circle, there must exist a force $\vec{\mathbf{F}}$ acting on the object directed towards the center.

In 1666, twenty years before Newton published his *Principia*, he realized that the moon is always "falling" towards the center of the earth; otherwise, by the First Law, it would continue in some linear trajectory rather than follow a circular orbit. Therefore there must be a *centripetal force*, a radial force pointing inward, producing this centripetal acceleration.

[1] Joni Mitchell, *The Circle Game,* Siquomb Publishing Company.

Because Newton's Second Law $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ is a vector equality, it can be applied to the radial direction to yield

$$F_r = m a_r. \tag{6.1.1}$$

6.2 Cylindrical Coordinate System

We first choose an origin and an axis we call the z-axis with unit vector $\hat{\mathbf{k}}$ pointing in the increasing z-direction. The level surface of points such that $z = z_P$ define a plane. We shall choose coordinates for a point P in the plane $z = z_P$ as follows.

One coordinate, r, measures the distance from the z-axis to the point P. The coordinate r ranges in value from $0 \le r \le \infty$. In Figure 6.1 we draw a few surfaces that have constant values of r. These `level surfaces' are circles.

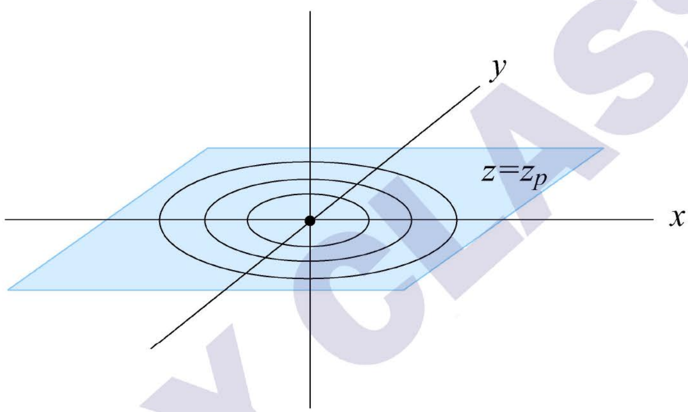


Figure 6.1 level surfaces for the coordinate r

Our second coordinate measures an angular distance along the circle. We need to choose some reference point to define the angle coordinate. We choose a 'reference ray', a horizontal ray starting from the origin and extending to $+\infty$ along the horizontal direction to the right. (In a typical Cartesian coordinate system, our 'reference ray' is the positive x-direction). We define the angle coordinate for the point P as follows. We draw a ray from the origin to the point P. We define the angle θ as the angle in the counterclockwise direction between our horizontal reference ray and the ray from the origin to the point P, (Figure 6.2). All the other points that lie on a ray from the origin to infinity passing through P have the same value as θ. For any arbitrary point, our angle coordinate θ can take on values from $0 \le \theta < 2\pi$. In Figure 6.3 we depict other `level surfaces', which are lines in the plane for the angle coordinate. The coordinates (r,θ) in the plane $z = z_P$ are called ***polar coordinates***.

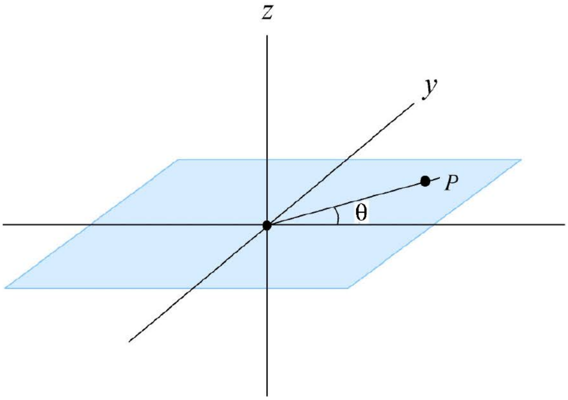


Figure 6.2 the angle coordinate

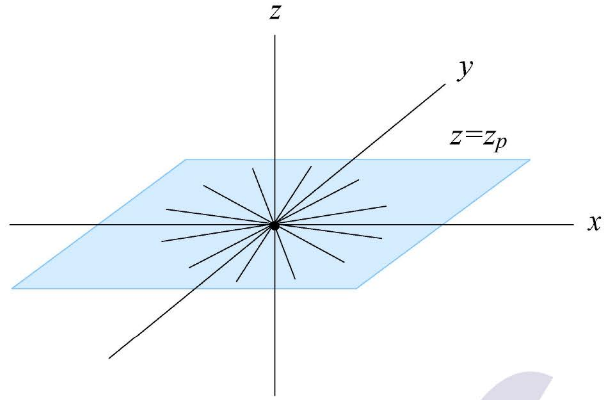


Figure 6.3 Level surfaces for the angle coordinate

6.2.1 Unit Vectors

We choose two unit vectors in the plane at the point P as follows. We choose $\hat{\mathbf{r}}$ to point in the direction of increasing r, radially away from the z-axis. We choose $\hat{\boldsymbol{\theta}}$ to point in the direction of increasing θ. This unit vector points in the counterclockwise direction, tangent to the circle. Our complete coordinate system is shown in Figure 6.4. This coordinate system is called a 'cylindrical coordinate system'. Essentially we have chosen two directions, radial and tangential in the plane and a perpendicular direction to the plane. If we are given polar coordinates (r,θ) of a point in the plane, the Cartesian coordinates (x,y) can be determined from the coordinate transformations

$$x = r\cos\theta\,, \tag{6.2.1}$$

$$y = r\sin\theta\,. \tag{6.2.2}$$

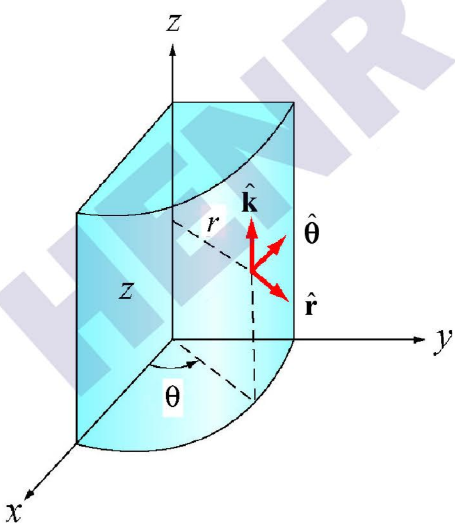


Figure 6.4 Cylindrical coordinates

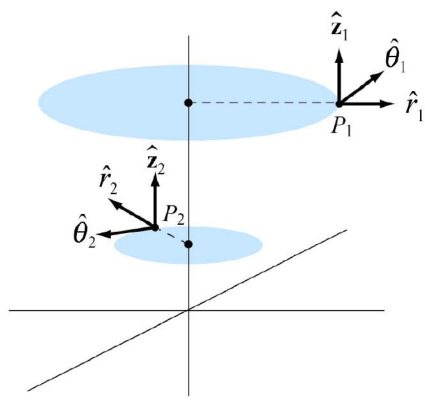


Figure 6.5 Unit vectors at two different points in polar coordinates.

Conversely, if we are given the Cartesian coordinates (x, y), the polar coordinates (r,θ) can be determined from the coordinate transformations

$$r = +(x^2 + y^2)^{1/2}, \tag{6.2.3}$$

$$\theta = \tan^{-1}(y / x). \tag{6.2.4}$$

Note that $r \geq 0$ so we always need to take the positive square root. Note also that $\tan\theta = \tan(\theta+\pi)$. Suppose that $0 \leq \theta \leq \pi/2$, then $x \geq 0$ and $y \geq 0$. Then the point $(-x,-y)$ will correspond to the angle $\theta+\pi$.

The unit vectors also are related by the coordinate transformations

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}, \tag{6.2.5}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \tag{6.2.6}$$

Similarly

$$\hat{\mathbf{i}} = \cos\theta\,\hat{\mathrm{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}, \tag{6.2.7}$$

$$\hat{\mathbf{j}} = \sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}. \tag{6.2.8}$$

One crucial difference between polar coordinates and Cartesian coordinates involves the choice of unit vectors. Suppose we consider a different point S in the plane. The unit vectors in Cartesian coordinates $(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S)$ at the point S have the same magnitude and point in the same direction as the unit vectors $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P)$ at P. Any two vectors that are equal in magnitude and point in the same direction are equal; therefore

$$\hat{\mathbf{i}}_S = \hat{\mathbf{i}}_P, \quad \hat{\mathbf{j}}_S = \hat{\mathbf{j}}_P. \tag{6.2.9}$$

A Cartesian coordinate system is the unique coordinate system in which the set of unit vectors at different points in space are equal. In polar coordinates, the unit vectors at two different points are not equal because they point in different directions. We show this in Figure 6.5.

6.2.2 Infinitesimal Line, Area, and Volume Elements in Cylindrical Coordinates

Consider a small infinitesimal displacement $d\vec{\mathbf{s}}$ between two points P_1 and P2 (Figure 6.6). This vector can be decomposed into

$$d\vec{\mathbf{s}} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\mathbf{k}}. \tag{6.2.10}$$

Consider an infinitesimal area element on the surface of a cylinder of radius r (Figure 6.7). The area of this element has magnitude

$$dA = r d\theta dz \,. \tag{6.2.11}$$

Area elements are actually vectors where the direction of the vector $d\vec{\mathbf{A}}$ points perpendicular to the plane defined by the area. Since there is a choice of direction, we shall choose the area vector to always point outwards from a closed surface. So for the surface of the cylinder, the infinitesimal area vector is

$$d\vec{\mathbf{A}} = r d\theta dz \,\hat{\mathbf{r}} \,. \tag{6.2.12}$$

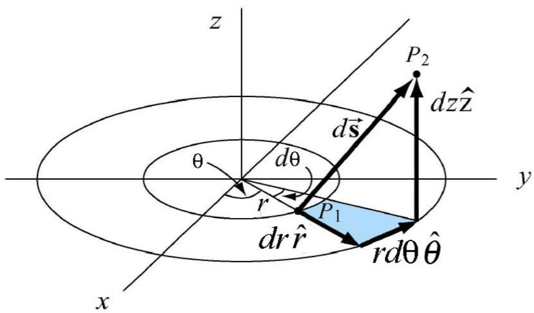


Figure 6.6 Displacement vector $d\vec{\mathbf{s}}$ between two points

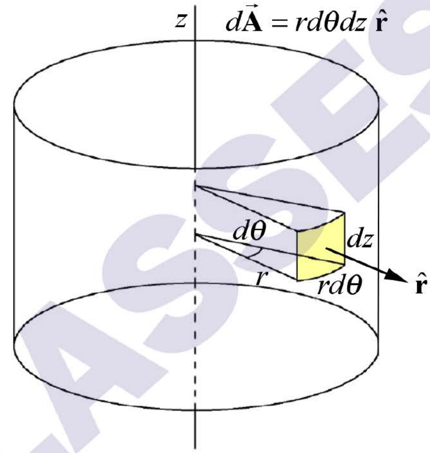


Figure 6.7 Area element for a cylinder: normal vector $\hat{\mathbf{r}}$

Example 6.1 Area Element of Disk

Consider an infinitesimal area element on the surface of a disc (Figure 6.8) in the xy-plane.

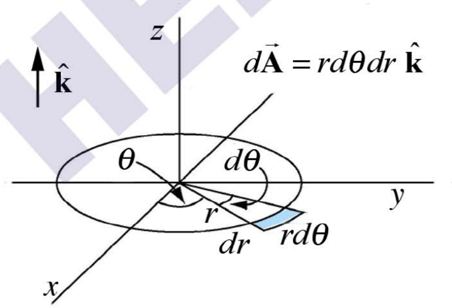


Figure 6.8 Area element for a disc: normal $\hat{\mathbf{k}}$

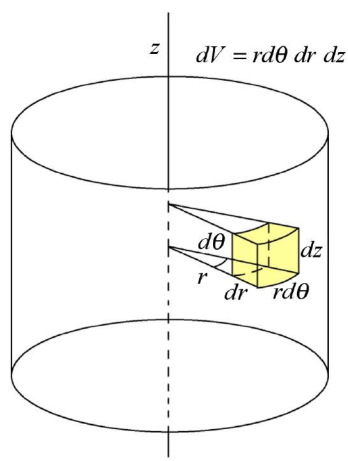


Figure 6.9 Volume element

Solution: The area element is given by the vector

$$d\vec{\mathbf{A}} = rd\theta dr\,\hat{\mathbf{k}}\,. \tag{6.2.13}$$

An infinitesimal volume element (Figure 6.9) is given by

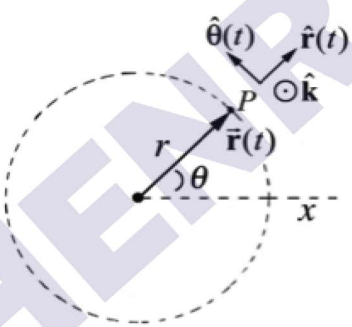
$$dV = rd\theta\; dr\; dz\,. \tag{6.2.14}$$

The motion of objects moving in circles motivates the use of the cylindrical coordinate system. This is ideal, as the mathematical description of this motion makes use of the radial symmetry of the motion. Consider the central radial point and a vertical axis passing perpendicular to the plane of motion passing through that central point. Then any rotation about this vertical axis leaves circles invariant (unchanged), making this system ideal for use for analysis of circular motion exploiting of the radial symmetry of the motion.

6.3 Circular Motion: Velocity and Angular Velocity

We can now begin our description of circular motion. In Figure 6.10 we sketch the position vector $\vec{\mathbf{r}}(t)$ of the object moving in a circular orbit of radius r. At time t, the particle is located at the point P with coordinates $(r, \theta(t))$ and position vector given by

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t)\,. \tag{6.3.1}$$



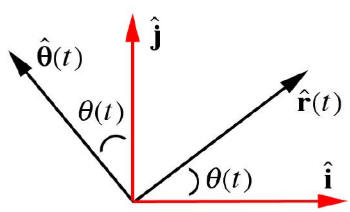


Figure 6.11 Unit vectors

Figure 6.10 A circular orbit.

At the point P, consider two sets of unit vectors ($\hat{\mathbf{r}}(t)$, $\hat{\boldsymbol{\theta}}(t)$) and ($\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$). In Figure 6.11 we see that a vector decomposition expression for $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is given by

$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\,, \tag{6.3.2}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}. \qquad (6.3.3)$$

We can write the position vector as

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t) = r(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}). \qquad (6.3.4)$$

The velocity is then

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = r\frac{d}{dt}(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}) = r(-\sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{i}} + \cos\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}}), \qquad (6.3.5)$$

where we used the chain rule to calculate that

$$\frac{d}{dt}\cos\theta(t) = -\sin\theta(t)\frac{d\theta(t)}{dt}\ , \qquad (6.3.6)$$

$$\frac{d}{dt}\sin\theta(t) = \cos\theta(t)\frac{d\theta(t)}{dt}\ . \qquad (6.3.7)$$

We now rewrite Eq. (6.3.5) as

$$\vec{\mathbf{v}}(t) = r\frac{d\theta(t)}{dt}(-\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}). \qquad (6.3.8)$$

Finally we substitute Eq. (6.3.3) into Eq. (6.3.8) and obtain an expression for the velocity of a particle in a circular orbit

$$\vec{\mathbf{v}}(t) = r\frac{d\theta(t)}{dt}\hat{\boldsymbol{\theta}}(t). \qquad (6.3.9)$$

We denote the rate of change of angle with respect to time by the Greek letter ω,

$$\omega \equiv \frac{d\theta}{dt}, \qquad (6.3.10)$$

which can be positive (counterclockwise rotation in Figure 6.10), zero (no rotation), or negative (clockwise rotation in Figure 6.10). This is often called the ***angular speed*** but it is actually the z-component of a vector called the ***angular velocity vector***.

$$\vec{\boldsymbol{\omega}} = \frac{d\theta}{dt}\,\hat{\mathbf{k}} = \omega\,\hat{\mathbf{k}}. \qquad (6.3.11)$$

The SI units of angular velocity are $[\mathrm{rad}\cdot\mathrm{s}^{-1}]$. Thus the velocity vector for circular motion is given by

$$\vec{\mathbf{v}}(t) = r\omega\,\hat{\boldsymbol{\theta}}(t) \equiv v_\theta\,\hat{\boldsymbol{\theta}}(t), \qquad (6.3.12)$$

where the $\hat{\boldsymbol{\theta}}$ -component of the velocity is given by

$$v_\theta = r\frac{d\theta}{dt}. \tag{6.3.13}$$

We shall call v_θ the ***tangential component of the velocity***.

6.3.1 Geometric Derivation of the Velocity for Circular Motion

Consider a particle undergoing circular motion. At time t, the position of the particle is $\vec{\mathbf{r}}(t)$. During the time interval Δt, the particle moves to the position $\vec{\mathbf{r}}(t+\Delta t)$ with a displacement $\Delta\vec{\mathbf{r}}$.

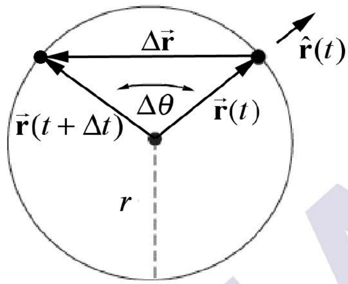


Figure 6.12 Displacement vector for circular motion

The magnitude of the displacement, $\left|\Delta\vec{\mathbf{r}}\right|$, is represented by the length of the horizontal vector $\Delta\vec{\mathbf{r}}$ joining the heads of the displacement vectors in Figure 6.12 and is given by

$$\left|\Delta\vec{\mathbf{r}}\right| = 2r\sin(\Delta\theta/2). \tag{6.3.14}$$

When the angle $\Delta\theta$ is small, we can approximate

$$\sin(\Delta\theta/2) \cong \Delta\theta/2. \tag{6.3.15}$$

This is called the *small angle approximation*, where the angle $\Delta\theta$ (and hence $\Delta\theta/2$) is measured in radians. This fact follows from an infinite power series expansion for the sine function given by

$$\sin\left(\frac{\Delta\theta}{2}\right) = \frac{\Delta\theta}{2} - \frac{1}{3!}\left(\frac{\Delta\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\Delta\theta}{2}\right)^5 - \cdots. \tag{6.3.16}$$

When the angle $\Delta\theta/2$ is small, only the first term in the infinite series contributes, as successive terms in the expansion become much smaller. For example, when $\Delta\theta/2 = \pi/30 \cong 0.1$, corresponding to 6°, $(\Delta\theta/2)^3/3! \cong 1.9\times10^{-4}$; this term in the power series is three orders of magnitude smaller than the first and can be safely ignored for small angles.

Using the small angle approximation, the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| \cong r\,\Delta\theta\,. \tag{6.3.17}$$

This result should not be too surprising since in the limit as $\Delta\theta$ approaches zero, the length of the chord approaches the arc length $r\Delta\theta$.

The magnitude of the velocity, $\left|\vec{\mathbf{v}}\right| \equiv v$, is then seen to be proportional to the rate of change of the magnitude of the angle with respect to time,

$$v \equiv \left|\vec{\mathbf{v}}\right| = \lim_{\Delta t\to 0}\frac{\left|\Delta\vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t\to 0}\frac{r\left|\Delta\theta\right|}{\Delta t} = r\lim_{\Delta t\to 0}\frac{\left|\Delta\theta\right|}{\Delta t} = r\left|\frac{d\theta}{dt}\right| = r\left|\omega\right|. \tag{6.3.18}$$

The direction of the velocity can be determined by considering that in the limit as $\Delta t \to 0$ (note that $\Delta\theta \to 0$), the direction of the displacement $\Delta\vec{\mathbf{r}}$ approaches the direction of the tangent to the circle at the position of the particle at time t (Figure 6.13).

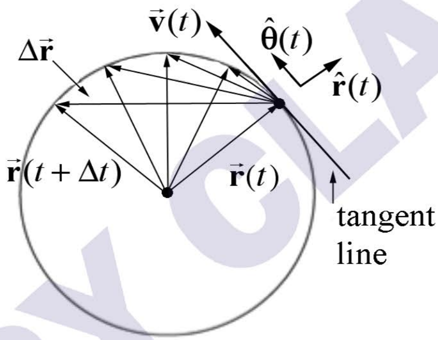


Figure 6.13 Direction of the displacement approaches the direction of the tangent line

Thus, in the limit $\Delta t \to 0$, $\Delta\vec{\mathbf{r}} \perp \vec{\mathbf{r}}$, and so the direction of the velocity $\vec{\mathbf{v}}(t)$ at time t is perpendicular to the position vector $\vec{\mathbf{r}}(t)$ and tangent to the circular orbit in the $+\hat{\boldsymbol{\theta}}$-direction for the case shown in Figure 6.13.

6.4 Circular Motion: Tangential and Radial Acceleration

When the motion of an object is described in polar coordinates, the acceleration has two components, the tangential component a_θ, and the radial component, a_r. We can write the acceleration vector as

$$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}}(t) + a_\theta\,\hat{\boldsymbol{\theta}}(t)\,. \tag{6.4.1}$$

Keep in mind that as the object moves in a circle, the unit vectors $\hat{\mathbf{r}}(t)$ and $\hat{\boldsymbol{\theta}}(t)$ change direction and hence are not constant in time.

We will begin by calculating the tangential component of the acceleration for circular motion. Suppose that the tangential velocity $v_\theta = r\omega$ is changing in magnitude due to the presence of some tangential force, where ω is the z-component of the angular velocity; we shall now consider that $\omega(t)$ is changing in time, (the magnitude of the velocity is changing in time). Recall that in polar coordinates the velocity vector Eq. (6.3.12) can be written as

$$\vec{\mathbf{v}}(t) = r\omega\,\hat{\boldsymbol{\theta}}(t)\,. \tag{6.4.2}$$

We now use the product rule to determine the acceleration.

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = r\frac{d\omega(t)}{dt}\hat{\boldsymbol{\theta}}(t) + r\omega(t)\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}\,. \tag{6.4.3}$$

Recall from Eq. (6.3.3) that $\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}$. So we can rewrite Eq. (6.4.3) as

$$\vec{\mathbf{a}}(t) = r\frac{d\omega(t)}{dt}\hat{\boldsymbol{\theta}}(t) + r\omega(t)\frac{d}{dt}(-\sin\theta(t)\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}})\,. \tag{6.4.4}$$

We again use the chain rule (Eqs. (6.3.6) and (6.3.7)) and find that

$$\vec{\mathbf{a}}(t) = r\frac{d\omega(t)}{dt}\hat{\boldsymbol{\theta}}(t) + r\omega(t)\left(-\cos\theta(t)\frac{d\theta(t)}{dt}\hat{\mathbf{i}} - \sin\theta(t)\frac{d\theta(t)}{dt}\,\hat{\mathbf{j}}\right)\,. \tag{6.4.5}$$

Recall that $\omega \equiv d\theta / dt$, and from Eq. (6.3.2), $\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}$, therefore the acceleration becomes

$$\vec{\mathbf{a}}(t) = r\frac{d\omega(t)}{dt}\,\hat{\boldsymbol{\theta}}(t) - r\omega^2(t)\,\hat{\mathbf{r}}(t)\,. \tag{6.4.6}$$

We denote the rate of change of ω with respect to time by the Greek letter α,

$$\alpha \equiv \frac{d\omega}{dt}\,, \tag{6.4.7}$$

which can be positive, zero, or negative. This is often called the ***angular acceleration*** but it is actually the z-component of a vector called the ***angular acceleration vector***.

$$\vec{\boldsymbol{\alpha}} = \frac{d\omega}{dt}\,\hat{\mathbf{k}} = \frac{d^2\theta}{dt^2}\,\hat{\mathbf{k}} \equiv \alpha\,\hat{\mathbf{k}}\,. \tag{6.4.8}$$

The SI units of angular acceleration are $[\text{rad} \cdot \text{s}^{-2}]$. The ***tangential component of the acceleration*** is then

$$a_\theta = r\alpha\,. \qquad (6.4.9)$$

The ***radial component of the acceleration*** is given by

$$a_r = -r\omega^2 < 0\ . \qquad (6.4.10)$$

Because $a_r < 0$, that radial vector component $\vec{\mathbf{a}}_r(t) = -r\omega^2\,\hat{\mathbf{r}}(t)$ is always directed towards the center of the circular orbit.

6.5 Period and Frequency for Uniform Circular Motion

If the object is constrained to move in a circle and the total tangential force acting on the object is zero, $F_\theta^{\text{total}} = 0$. By Newton's Second Law, the tangential acceleration is zero,

$$a_\theta = 0\,. \qquad (6.5.1)$$

This means that the magnitude of the velocity (the speed) remains constant. This motion is known as ***uniform circular motion***. The acceleration is then given by only the acceleration radial component vector

$$\vec{\mathbf{a}}_r(t) = -r\omega^2(t)\,\hat{\mathbf{r}}(t) \quad \text{uniform circular motion}\,. \qquad (6.5.2)$$

Since the speed $v = r|\omega|$ is constant, the amount of time that the object takes to complete one circular orbit of radius r is also constant. This time interval, T, is called the ***period.*** In one period the object travels a distance $s = vT$ equal to the circumference, $s = 2\pi r$; thus

$$s = 2\pi r = vT\,. \qquad (6.5.3)$$

The period T is then given by

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{r|\omega|} = \frac{2\pi}{|\omega|}. \qquad (6.5.4)$$

The ***frequency*** f is defined to be the reciprocal of the period,

$$f = \frac{1}{T} = \frac{|\omega|}{2\pi}. \qquad (6.5.5)$$

The SI unit of frequency is the inverse second, which is defined as the hertz, $\left[\mathrm{s}^{-1}\right] \equiv [\mathrm{Hz}]$.

The magnitude of the radial component of the acceleration can be expressed in several equivalent forms since both the magnitudes of the velocity and angular velocity are related by $v = r|\omega|$. Thus we have several alternative forms for the magnitude of the centripetal acceleration. The first is that in Equation (6.6.3). The second is in terms of the radius and the angular velocity,

$$|a_r| = r\omega^2 . \tag{6.5.6}$$

The third form expresses the magnitude of the centripetal acceleration in terms of the speed and radius,

$$|a_r| = \frac{v^2}{r}. \tag{6.5.7}$$

Recall that the magnitude of the angular velocity is related to the frequency by $|\omega| = 2\pi f$, so we have a fourth alternate expression for the magnitude of the centripetal acceleration in terms of the radius and frequency,

$$|a_r| = 4\pi^2 r f^2 . \tag{6.5.8}$$

A fifth form commonly encountered uses the fact that the frequency and period are related by $f = 1/T = |\omega|/2\pi$. Thus we have the fourth expression for the centripetal acceleration in terms of radius and period,

$$|a_r| = \frac{4\pi^2 r}{T^2} . \tag{6.5.9}$$

Other forms, such as $4\pi^2 r^2 f / T$ or $2\pi r\omega f$, while valid, are uncommon.

Often we decide which expression to use based on information that describes the orbit. A convenient measure might be the orbit's radius. We may also independently know the period, or the frequency, or the angular velocity, or the speed. If we know one, we can calculate the other three but it is important to understand the meaning of each quantity.

6.5.1 Geometric Interpretation for Radial Acceleration for Uniform Circular Motion

An object traveling in a circular orbit is always accelerating towards the center. Any radial inward acceleration is called *centripetal acceleration*. Recall that the direction of the velocity is always tangent to the circle. Therefore the direction of the velocity is constantly changing because the object is moving in a circle, as can be seen in Figure 6.14. Because the velocity changes direction, the object has a nonzero acceleration.

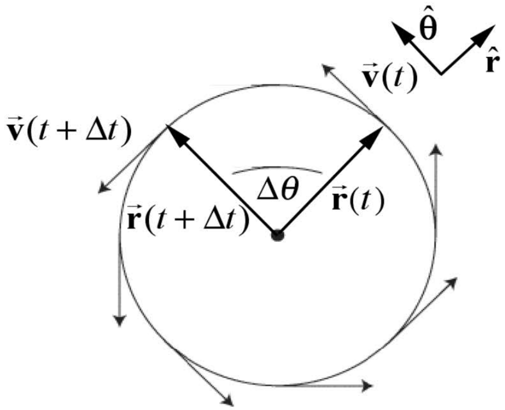


Figure 6.14 Direction of the velocity for circular motion.

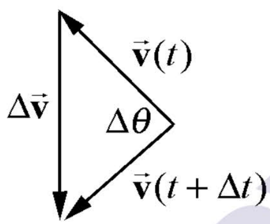


Figure 6.15 Change in velocity vector.

The calculation of the magnitude and direction of the acceleration is very similar to the calculation for the magnitude and direction of the velocity for circular motion, but the change in velocity vector, $\Delta\vec{\mathbf{v}}$, is more complicated to visualize. The change in velocity $\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$ is depicted in Figure 6.15. The velocity vectors have been given a common point for the tails, so that the change in velocity, $\Delta\vec{\mathbf{v}}$, can be visualized. The length $|\Delta\vec{\mathbf{v}}|$ of the vertical vector can be calculated in exactly the same way as the displacement $|\Delta\vec{\mathbf{r}}|$. The magnitude of the change in velocity is

$$|\Delta\vec{\mathbf{v}}| = 2v\sin(\Delta\theta/2). \tag{6.6.1}$$

We can use the small angle approximation $\sin(\Delta\theta/2) \cong \Delta\theta/2$ to approximate the magnitude of the change of velocity,

$$|\Delta\vec{\mathbf{v}}| \cong v|\Delta\theta|. \tag{6.6.2}$$

The magnitude of the radial acceleration is given by

$$|a_r| = \lim_{\Delta t\to 0}\frac{|\Delta\vec{\mathbf{v}}|}{\Delta t} = \lim_{\Delta t\to 0}\frac{v|\Delta\theta|}{\Delta t} = v\lim_{\Delta t\to 0}\frac{|\Delta\theta|}{\Delta t} = v\left|\frac{d\theta}{dt}\right| = v|\omega|. \tag{6.6.3}$$

The direction of the radial acceleration is determined by the same method as the direction of the velocity; in the limit $\Delta\theta \to 0$, $\Delta\vec{\mathbf{v}} \perp \vec{\mathbf{v}}$, and so the direction of the acceleration radial component vector $\vec{\mathbf{a}}_r(t)$ at time t is perpendicular to position vector $\vec{\mathbf{v}}(t)$ and directed inward, in the $-\hat{\mathbf{r}}$-direction.