1.1 General Equation

$ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of second degree that represents two straight line passing through the origin. The individual equations of these straight lines can be obtained by factorising the expres sion $ax^2 + 2hxy + by^2 = 0$ into two real linear factors. The equation represents two lines in real XY plane only if $h^2 - ab \geq 0$.

1.2. Slopes of the lines

$$ax^2 + 2hxy + by^2 = 0$$

Let m_1, m_2 be the slopes of the lines through origin expressed by

$$ax^2 + 2hxy + by^2 = 0 \qquad \text{. . . (i)}$$

$\Rightarrow$ The lines are $y = m_1x$ and $y = m_2x$ and the equation of pair is $(y - m_1x)(y - m_2x) = 0$

$\Rightarrow$ $m_1m_2x^2 - xy(m_1 + m_2) + y^2 = 0 \qquad \text{. . . (ii)}$

$$\Rightarrow \quad \frac{m_1m_2}{a} = \frac{m_1 + m_2}{-2h} = \frac{1}{b}$$

$$\Rightarrow \quad m_1 + m_2 = \frac{-2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

We can solve for m_1 and m_2 to get

$$\left[m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \quad \text{and} \quad m_2 = \frac{-h - \sqrt{h^2 - ab}}{b} \right]$$

Note : Again that $h^2 - ab \geq 0$ is the necessary condition for the equation representing real lines.

Illustration - 1 *What does the equation $x^2 - 5xy + 4y^2 = 0$ represents?*

SOLUTION :

$$x^2 - 5xy + 4y^2 = 0$$

$\Rightarrow$ $x^2 - 4xy - xy + 4y^2 = 0 \quad \Rightarrow \quad (x - 4y)(x - y) = 0$

$\Rightarrow$ The equation represent two straight lines through origin whose Equations are :

$x - 4y = 0$ and $x - y = 0$.

1.3 Angle between the lines

$ax^2 + 2hxy + by^2 = 0$

If the acute angle between lines $ax^2 + 2hxy + by^2 = 0$

then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right| = \dfrac{\sqrt{\dfrac{4h^2}{b^2} - \dfrac{4a}{b}}}{|1 + a/b|}$

$\Rightarrow \quad \theta = \tan^{-1}\left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$

Hence we also have following results.

(i) The lines $ax^2 + 2hxy + by^2 = 0$ are mutually perpendicular if and only if $a + b = 0$.
i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

(ii) The lines $ax^2 + 2hxy + by^2 = 0$ are coincident if $h^2 - ab = 0$.

1.4 Angle bisector of lines

$ax^2 + 2hxy + by^2 = 0.$

Let m_1, m_2 be the slopes of lines $ax^2 + 2hxy + by^2 = 0$.

$\Rightarrow$ lines are $y - m_1 x = 0$ and $y - m_2 x = 0$ $\quad [m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b]$

$\Rightarrow$ bisectors of angles are :

$$\frac{y - m_1x}{\sqrt{1+m_1^2}} = \pm\left(\frac{y - m_2x}{\sqrt{1+m_2^2}}\right) \Rightarrow \left(1 + m_2^2\right)(y - m_1x)^2 - \left(1 + m_1^2\right)(y - m_2x)^2 = 0$$

On simplification we get :

$-y^2(m_1 + m_2) + x^2(m_1 + m_2) - 2xy(1 - m_1m_2) = 0$

$\Rightarrow \quad x^2 - y^2 = \dfrac{2xy(1 - m_1m_2)}{m_1 + m_2}$

$\Rightarrow \quad x^2 - y^2 = \dfrac{2xy(1 - a/b)}{-2h/b}$

$\Rightarrow$ The equation of Bisectors is $\dfrac{x^2 - b^2}{a - b} = \dfrac{xy}{h}$.

1.5 Pair of lines perpendicular to the lines

Let L_1 and L_2 be the lines $ax^2 + 2hxy + by^2 = 0$.

Let P_1 be the line Perpendicular to L_1 and P_2 be the line perpendicular to L_2.

We have to find equation of P_1P_2.

Let L_1 be $y - m_1x = 0$ and L_2 be $y - m_2x = 0$

$\Rightarrow$ P_1 is $m_1y + x = 0$ and P_2 is $m_2y + x = 0$

$\Rightarrow$ Pair P_1P_2 is $(m_1y + x) . (m_2y + x) = 0$

$\Rightarrow$ $m_1 m_2 y^2 + xy (m_1 + m_2) + x^2 = 0$

$$\Rightarrow \frac{a}{b}y^2 + xy\left(-\frac{2h}{b}\right) + x^2 = 0$$

$\Rightarrow$ $bx^2 - 2hxy + ay^2 = 0$ is the equation of the pair of lines perpendicular to the pairs of lines $ax^2 + 2hxy + by^2 = 0$.

Note : By interchanging the coefficients of x^2 and y^2 and reversing the sign of the xy term, we can get equation of P_1P from L_1L_2.

Illustration - 2 *Find the area formed by the triangle whose sides are $y^2 - 9xy + 18x^2 = 0$ and $y = 9$.*

SOLUTION :

$y^2 - 9xy + 18x^2 = 0$

$\Rightarrow$ $(y - 3x)(y - 6x) = 0$

$\Rightarrow$ The sides of the triangle are : $y - 3x = 0$ and $y - 6x = 0$ and $y - 9 = 0$.

By solving these simultaneousy, we get the vertices as :

$A \equiv (0, 0)$; $B \equiv (3/2, 9)$; $C \equiv (3, 9)$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{3}{2} & 9 & 1 \\ 3 & 9 & 1 \end{vmatrix} = \frac{27}{4} \text{ sq. units.}$$

Illustration - 3 *Find the angle between the lines $x^2 + 4y^2 - 7xy = 0$.*

SOLUTION :

$$\theta = \tan^{-1}\frac{2\sqrt{h^2 - ab}}{a+b} = \tan^{-1}\left[\frac{2\sqrt{\left(\frac{-7}{2}\right) - 1(4)}}{1+4}\right] = \tan^{-1}\left[\frac{\sqrt{33}}{5}\right].$$

Illustration - 4 *Find the equation of pair of lines through origin which form an equilateral triangle with the lines $Ax + By + C = 0$. Also find the area of this equilateral triangle.*

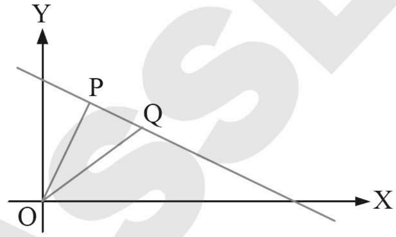
SOLUTION :

Let PQ be the side of the equilateral triangle lying on the line $Ax + By + C = 0$. Let m be the slope of the line through origin and making an angle of 60° with $Ax + By + C = 0$.

$\Rightarrow$ m is the slopes of OP or OQ.

As the triangle is equilateral, the line $Ax + By + C = 0$ makes an angle 60° with OP and OQ.

$$\Rightarrow \tan 60^\circ = \left|\frac{m-(-A/B)}{1+m\left(\frac{-A}{B}\right)}\right|$$



$$\Rightarrow 3 = \left(\frac{mB + A}{B - mA}\right)^2 \qquad \text{. . . (i)}$$

This quadratic will give two values of m which are slope of OP and OQ. As OP and OQ pass through origin, their equations can be taken as :

$$y = mx \qquad \text{. . . (ii)}$$

As we have to find the equation of OP and OQ, we will not find values of m but we will eliminate m between (i) and (ii) to directly get the equation of the pairs of lines : OP and OQ.

$$\Rightarrow \left(\frac{By/x + A}{B - yA/x}\right)^2 \qquad \Rightarrow 3 = \left(\frac{By + Ax}{Bx - yA}\right)^2$$

$\Rightarrow$ $3\,(B^2x^2 + y^2A^2 - 2ABxy) = (B^2x^2 + A^2x^2 + 2ABxy)$

$\Rightarrow$ $(A^2 - 3B^2)\,x^2 + 8ABxy + (B^2 - 3A^2)\,y^2 = 0$ is the pair of lines through origin making an equilateral triangle (OPQ) with $Ax + By + C = 0$.

$$\text{Area of } \Delta OPQ = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}\left(\frac{P}{\sin 60}\right)^2 \quad [\text{where } P = \text{altitude}]$$

$$\Rightarrow \text{area} = \frac{\sqrt{3}}{4} \times \frac{4}{3}P^2 = \frac{1^2}{\sqrt{3}}P^2 = \frac{1}{\sqrt{3}}\left[\frac{|C|}{\sqrt{A^2 + B^2}}\right]^2 = \frac{C^2}{\sqrt{3}\,(A^2 + B^2)}$$

Illustration - 5 *If a pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ is such that each pair bisects the angle between the other pair, prove that $pq = -1$.*

SOLUTION :

The pair of bisectors for $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2-y^2}{1-(-1)}=\frac{xy}{-p}$$

$\Rightarrow \quad x^2-y^2=\frac{2xy}{-p} \qquad \Rightarrow \quad x^2+\frac{2}{p}xy-y^2=0$

As $x^2+\frac{2}{p}xy-y^2=0$ and $x^2-2qxy-y^2=0$ coincide,

we have :

$\frac{1}{1}=\frac{2/p}{-2q}=\frac{-1}{-1} \qquad \Rightarrow \qquad \frac{2}{p}=-2q$

$\Rightarrow \quad pq=-1.$

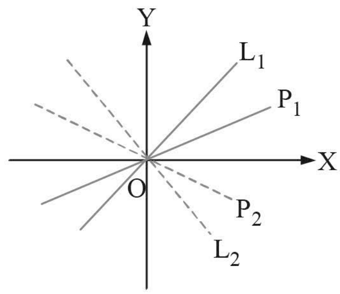
Illustration - 6 *Prove that the angle between one of the lines given by $ax^2+2hxy+by^2=0$ and one of the lines $ax^2+2hxy+by^2+\lambda(x^2+y^2)=0$ is equal to the angle between the other two lines of the system.*

SOLUTION :

Let L_1L_2 be one pair and P_1P_2 be other, if the angle between L_1P_1 is equal to the angle between L_2P_2, the pair of bisectors of L_1L_2 is same as that of P_1P_2.

$\Rightarrow$ Pair of bisectors of P_1P_2 is :

$$\frac{x^2-y^2}{(a+\lambda)-(b+\lambda)}=\frac{xy}{h} \qquad \Rightarrow \qquad \frac{x^2-y^2}{a-b}=\frac{xy}{h}$$



This is same as the bisector pair of L_1L_2.

Hence the statement is proved.

Illustration - 7 *Show that the orthocentre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$ is given by $\frac{x}{l}=\frac{y}{m}=\frac{a+b}{am^2-2hlm+bl^2}$.*

SOLUTION :

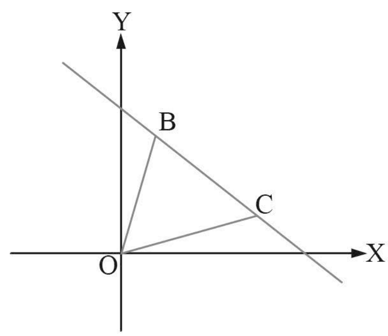
Let the triangle be OBC where O is origin and BC is the line $lx+my=1$.

$\Rightarrow \quad OB, OC$ are $ax^2+2hxy+by^2=0$.

The altitude from O to BC is $y-0=\frac{m}{l}(x-0) \quad \Rightarrow \quad mx-ly=0$

Let $OB : y-m_1x=0$ and $OC : y-m_2x=0$.

$\Rightarrow \quad B\equiv\left[\frac{1}{l+mm_1},\frac{m_1}{l+mm_1}\right]$



Slope of altitude from B to OC is $-1/m_2$

$\Rightarrow$ The equation of altitude from B is

$$y-\frac{m_1}{l+mm_1}=\frac{-1}{m_2}\left[x-\frac{1}{l+mm_1}\right]$$

$\Rightarrow$ $(l+mm_1)x+m_2(l+mm_1)y-(1+m_1m_2)=0$ . . . (i)

$\Rightarrow$ $mx-ly+0=0$ . . . (ii)

Solving (i) and (ii), we get orthcentre :

$$\frac{x}{-l(1+mm_1)}=\frac{y}{-m(1+m_1m_2)}=\frac{1}{-l(l+mm_1)-mm_2(l+mm_1)}$$

$$\Rightarrow \frac{x}{l}=\frac{y}{m}=\frac{-(1+a/b)}{-l^2-m^2m_1m_2-lm(m_1+m_2)}$$

$$\Rightarrow \frac{x}{l}=\frac{y}{m}=\frac{a+b}{bl^2+am^2-2hlm}$$ [using values of m_1m_2 and m_1+m_2]