Differential Calculus - 1

CONTINUITY

3.1 Introduction

Consider the functions : $f(x) = [x]$, $g(x) = x^2$, $x \in R$. The graphs of $f(x)$ and $g(x)$ in the neighbourhood of argument $x = 2$ are shown in fig. 1 & 2 respectively. There is a break in the graph of $f(x)$ at $x = 2$, whereas this is not so in that of $g(x)$. We express this difference by saying that the function $f(x)$ is discontinuous at $x = 2$ and the function $g(x)$ is continuous at $x = 2$. As we approach 2 from either left or right, the values of $g(x)$ approach its value at $x = 2$. But this does not happen for $f(x)$, and this brings about the break in its graph at $x = 2$.

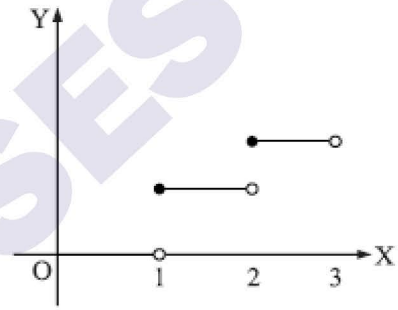


Fig. 1

OR mathematically,

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x) \quad \text{and} \quad \lim_{x\to 2^-} g(x) = \lim_{x\to 2^+} g(x) = g(2)$$

We are thus led to the following definitions :

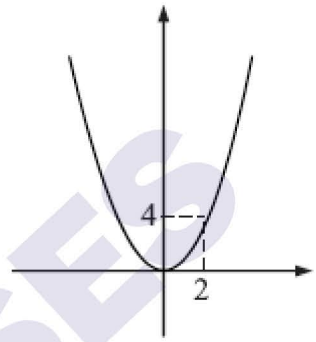


Fig. 2

(a) Continuity at $x = a$

A function $y = f(x)$ is continuous at $x = a$, if its limit at $x = a$ exists and is equal to $f(a)$ i.e.

Left hand limit = Right hand limit = $f(a)$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

(b) Discontinuity at $x = a$

We say that $f(x)$ is discontinuous at $x = a$, if $f(x)$ is not continuous at $x = a$.

OR

in other words, if any one or more of the conditions for the function $f(x)$ to be continuous fails to be satisfied, $f(x)$ is said to be discontinuous at $x = a$. Geometrically speaking, there must be a break in the graph of $f(x)$ at $x = a$.

3.2 Theorems on Continuity

1. Let $f(x)$ and $g(x)$ be the continuous functions $x = a$. Then the following functions are continuous at $x = a$.

   (a) $f(x) \pm g(x)$ **(b)** $k f(x), k g(x)$ [where k is real]

   (c) $f(x) \cdot g(x)$ **(d)** $\dfrac{f(x)}{g(x)}$, provided $g(a) \neq 0$.

2. $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n$ is the nth degree polynomial function. This function is continuous at all values of x.

3. $y = \sin x$, $y = \cos x$ are continuous for all x.

   $y = \log_a x$ is continuous for all $x > 0$.

   $y = a^x$ is continuous for all x.

4. If $y = f(x)$ is continuous for $x \in [a, b]$ and N is any number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c) = N$.

5. If $y = f(x)$ is continuous for $x \in [a, b]$ and $f(a)$ and $f(b)$ are of opposite signs, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b).

6. The Sandwich Theorem :

Suppose that $f(x) \le g(x) \le h(x)$

for all $x \ne c$ in some interval about c, and that $f(x)$ and $h(x)$ approach the same limit

L as x approaches c i.e. $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} h(x) = L$. Then $\lim\limits_{x \to c} g(x) = L$.

7. If the function f is continuous at $x = a$ and g is continuous at $x = f(a)$ then composite function $g\{f(x)\}$ is continuous at $x = a$.

Illustrating the Concepts :

(i) *Discuss the continuity of* $f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1} & ; \quad x \ne 0 \\ 0 & ; \quad x = 0 \end{cases}$ *at the point* $x = 0$.

$$\text{LHL} = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{\to 0 - 1}{\to 0 + 1} = -1$$

$$\text{RHL} = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

Divide N and D by $e^{1/h}$ to get :

$$\text{RHL} = \lim_{h \to 0} \frac{1 - \dfrac{1}{e^{1/h}}}{1 + \dfrac{1}{e^{1/h}}} = \frac{1 - (\to 0)}{1 + (\to 0)} = 1$$

$\Rightarrow$ L.H.L. $\ne$ R.H.L. $\Rightarrow$ $f(x)$ is discontinuous at $x = 0$.

(ii) *Discuss the continuity of the function :* $g(x) = [x] + [-x]$ *at integral values of* x.

Let us simplify the definition of the function :

(I) If x is an integer :

$[x] = x$ and $[-x] = -x$

$\Rightarrow \quad g(x) = x - x = 0$

(II) If x is not integer :

Let $x = n + f$

[where n is an integer and $f \in (0, 1)$]

$\Rightarrow \quad [x] = [n + f] = n$

and $[-x] = [-n - f] = [(-n - 1) + (1 - f)]$

$= -n - 1$

[because $0 < f < 1 \Rightarrow 0 \ (1 - f) < 1$]

Hence $g(x) = [x] + [-x]$

$= n + (-n - 1) = -1$

So we get :

$$g(x)=\begin{cases}0, \text{ if } x \text{ is an integer}\\ -1, \text{ if } x \text{ is not an integer}\end{cases}$$

Let us discuss the continuity of $g(x)$ at a point $x = a$ [where $a \in I$]

$$\text{L.H.L.} = \lim_{x\to a^-} g(x) = -1$$

[$\because$ as $x \to a$, x is not an integer]

$$\text{R.H.L.} = \lim_{x\to a^+} g(x) = -1$$

[$\because$ as $x \to a$, x is not an integer]

but $g(a) = 0$ because a is an integer. Hence $g(x)$ has a removable discontinuity at integral values of x.

Illustration - 31 *The values a and b so that the function :*

$$f(x)=\begin{cases} x + a\sqrt{2}\sin x & ; \quad 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & ; \quad \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & ; \quad \frac{\pi}{2} < x \le \pi \end{cases}$$

is continuous $x \in [0, \pi]$ *is :*

(A) $a=\frac{\pi}{6}, b=\frac{-\pi}{12}$ **(B)** $a=\frac{\pi}{3}, b=\frac{-\pi}{12}$ **(C)** $a=\frac{\pi}{6}, b=\frac{\pi}{12}$ **(D)** *None of these*

SOLUTION : (A)

At $x = \pi/4$: Left hand limit $= \lim_{x\to\frac{\pi}{4}^-} f(x) = \lim_{x\to\frac{\pi}{4}^-} (x + a\sqrt{2}\sin x) = \frac{\pi}{4} + a$

Right hand limit $= \lim_{x\to\frac{\pi}{4}^+} f(x) = \lim_{x\to\frac{\pi}{4}^+} (2x\cot x + b) = \frac{\pi}{2} + b$

$$f\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right)\cot\frac{\pi}{4} + b = \frac{\pi}{2} + b$$

for continuity, these three must be equal

$$\Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \quad \Rightarrow \quad a - b = \frac{\pi}{4} \qquad \text{. . . (i)}$$

At $x = \pi/2$: Left hand limit $= \lim_{x\to\frac{\pi}{2}^-} (2x\cot x + b) = 0 + b = b$

Right hand limit = $\lim_{x \to \frac{\pi}{2}^+} (a\cos 2x - b\sin x) = -a - b$ $\quad f\left(\frac{\pi}{2}\right) = 0 + b$

for continuity, $b = -a - b \Rightarrow a + 2b = 0$ . . . **(ii)**

Solving (i) and (ii) for a and b, we get : $b = -\frac{\pi}{12}, \quad a = \frac{\pi}{6}$

Illustration - 32

$$Let\ f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & ; \ \frac{-\pi}{6} < x < 0 \\ b & ; \ x = 0 \\ e^{\left(\frac{\tan 8x}{\tan 3x}\right)} & ; \ 0 < x < \frac{\pi}{6} \end{cases}$$

The value a and b such that f (x) is continuous at x = 0 is :

(A) $a = 8, b = e^8$ **(B)** $a = \frac{8}{3}, b = e^{-8}$ **(C)** $a = \frac{8}{3}, b = e^{8/3}$ **(D)** *None of these*

SOLUTION : (C)

Left hand limit at $x = 0$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left[(1+|\sin x|)^{\frac{a}{|\sin x|}}\right]$$

$$\Rightarrow \quad \text{L.H.L} = \lim_{h \to 0} f(0-h)$$

$$\Rightarrow \text{L.H.L} = \lim_{h \to 0} \left[(1+|\sin h|)^{\frac{a}{|\sin h|}}\right] = e^a$$

$$\left[\text{using} : \lim_{t \to 0} (1+t)^{1/t} = e\right]$$

Right hand limit $x = 0$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\frac{\tan 8x}{\tan 3x}}$$

$$\Rightarrow \quad \text{R.H.L.} = \lim_{h \to 0} f(0+h)$$

$$\Rightarrow \quad \text{R.H.L.} = \lim_{h \to 0} e^{\frac{\tan 8h}{\tan 3h}}$$

$$\Rightarrow \text{R.H.L.} = \lim_{h \to 0} e^{\frac{8}{3}\left(\frac{\tan 8h}{8h} \cdot \frac{3h}{\tan 3h}\right)} = e^{8/3}$$

for continuity,

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$$\Rightarrow \quad e^a = e^{2/3} = b \quad \Rightarrow \quad a = \frac{8}{3}, b = e^{8/3}$$

Illustration - 33

Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & ; \quad x<0\\ a & ; \quad x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; \quad x>0\end{cases}$

The value of a, if possible, so that the function is continuous at x = 0 is :

(A) 6 **(B)** 8 **(C)** – 6 **(D)** *None of these*

SOLUTION : (B)

It is given that

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & ; \quad x<0\\ a & ; \quad x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ; \quad x>0\end{cases}$$

is continuous at $x = 0$.

So we can take :

$$\lim_{x\to 0^-} f(x)=f(0)=\lim_{x\to 0^+} f(x)$$

Left hand limit at $x = 0$,

$$\text{L.H.L.}=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1-\cos 4x}{x^2}$$

Now, $\text{L.H.L.} = \lim_{h\to 0} f(0-h)$

$$\Rightarrow \quad \text{L.H.L.}=\lim_{h\to 0}\frac{1-\cos 4h}{h^2}=\lim_{h\to 0}\frac{2\sin^2 2h}{h^2}=8$$

$\left[\text{using}: \lim_{t\to 0}\frac{\sin t}{t}=1\right]$

Right hand limit at $x = 0$

$$\text{R.H.L.} = \lim_{h\to 0} f(0+h)$$

$$\Rightarrow \quad \text{R.H.L.} = \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}$$

Rationalise denominator to get :

$$\text{R.H.L.} = \lim_{h\to 0}\frac{\sqrt{h}}{\sqrt{h}}\left(\sqrt{16+\sqrt{h}}+4\right)=8$$

For function $f(x)$ to be continuous at $x = 0$,

$$\text{L.H.L.} = \text{R.H.L.} = f(0)$$

$\Rightarrow \quad 8 = 8 = a$

$\Rightarrow \quad a = 8$

Illustration - 34

If $f(x) = \dfrac{\sin 2x + A\sin x + B\cos x}{x^3}$ *is continuous at x = 0, find the values of A, B and f (0) are :*

(A) $A = 2,\ B = 0,\ f(0) = -1$ (B) $A = -2,\ B = 0,\ f(0) = -1$

(C) $A = 0,\ B = -2,\ f(0) = 1$ (D) *None of these*

SOLUTION : (B)

As $f(x)$ is continuous at $x = 0$, $f(0) = \lim_{x \to 0} \dfrac{\sin 2x + A\sin x + B\cos x}{x^3}$

Using expansions of $sin2x$, $sinx$ and $cosx$, we get :

$$f(0) = \lim_{x \to 0} \frac{\left(2x - \frac{(2x)^3}{\lfloor 3} + \dots\right) + A\left(x - \frac{(x)^3}{\lfloor 3} + \dots\right) + B\left(1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots\right)}{x^3}$$

$$f(0) = \lim_{x \to 0} \frac{B + (A+2)x\ \frac{-B}{2}x^2 - \left[\frac{2^3}{\lfloor 3} + \frac{A}{\lfloor 3}\right]x^3 + \dots}{x^3}$$

For above limit to be finite (exist), coefficient of x^0, x^1 and x^2 should be 0 in numerator i.e.,

$$B = 0,\ A + 2 = 0 \quad \text{and} \quad \frac{-B}{2} = 0 \quad \Rightarrow \quad A = -2 \quad \text{and} \quad B = 0$$

On replacing, we get : $f(0) = \lim_{x \to 0} = \dfrac{-x^3 + \dots}{x^3}$

$f(0) = -1$

So, we get : $A = -2$, $B = 0$, $f(0) = -1$.

Note : We can also solve this question using L'Hospital rule.

Illustration - 35

The point where $f(x) = \lim_{n \to \infty} \left(\sin \dfrac{\pi x}{2}\right)^{2n}$ *is discontinuous are :*

(A) $x = n,\ n \in I$ (B) $x = 2n,\ n \in I$

(C) $x = (2n+1),\ n \in I$ (D) *None of these*

SOLUTION : (C)

Since $\lim_{n \to \infty} x^{2n} = \begin{cases} 0 & ; & |x| < 1 \\ 1 & ; & |x| = 1 \end{cases}$

$\therefore \quad f(x) = \lim_{n \to \infty} \left(\sin \frac{\pi x}{2} \right)^{2n}$

$= \begin{cases} 0 & ; & \left| \sin \frac{\pi x}{2} \right| < 1 \\ 1 & ; & \left| \sin \frac{\pi x}{2} \right| = 1 \end{cases}$

Thus $f(x)$ is continuous for all x, except for those values of x for which $\left| \sin \frac{\pi x}{2} \right| = 1$

$\Rightarrow \quad \sin \frac{\pi x}{2} = \pm 1$

$\Rightarrow \quad x = (2n + 1)\, \pi$

i.e. $\quad x$ is an odd integer

$\Rightarrow \quad x = (2n + 1) \qquad$ [where $n \in I$]

Check continuity at $x = (2n + 1)$:

L.H.L. $= \lim_{x \to 2n+1} f(x) = 0 \quad$ **. . . (i)**

and $f(2n + 1) = 1$

L.H.L. $\neq f(2n + 1)$,

$\Rightarrow \quad f(x)$ is discontinuous at $x = 2n + 1$ [i.e. at odd integers]

Hence $f(x)$ is discontinuous at $x = (2n + 1)$.

Illustration - 36 *The number of points where $f(x)$ is discontinuous in $[0, 2]$ where* $f(x) = \begin{cases} [\cos \pi x] & ; & x \le 1 \\ [x - 2]\,|2x - 3| & ; & x > 1 \end{cases}$ *where [ ] : represents the greatest integer function is :*

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

SOLUTION : (C)

First of all find critical points where $f(x)$ may be discontinuous.

Consider $x \in [0, 1]$: $\quad f(x)\ [\cos \pi x]$

x is discontinuous where $x \in I$. $\quad \Rightarrow \quad \cos \pi x \in I$.

In $[0, 1]$, $\cos \pi x$ is an integer at $x = 0$, $x = \frac{1}{2}$ and $x = 1$.

$\Rightarrow \quad x = 0, x = \frac{1}{2}$ and $x = 1$ are critical points $\quad$ **. . . (i)**

Consider $x \in (1, 2]$:

$f(x) = [x - 2]\,|2x - 3|$

In $\quad x \in (1, 2)$, $[x - 2] = -1 \quad$ and $\quad$ for $x = 2 \quad ; \quad [x - 2] = 0$

Also $|2x-3|$ $\Rightarrow$ $x=\dfrac{3}{2}$

$\Rightarrow$ $x=\dfrac{3}{2}$ and $x=2$ are critical points . . . **(ii)**

Combining (i) and (ii), critical points are $0, \dfrac{1}{2}, 1, \dfrac{3}{2}, 2$.

On dividing $f(x)$ about the 5 critical points, we get :

$$f(x)=\begin{cases} 1 & ; \quad x=0 \quad \because \quad \cos(\pi 0)=1 \\ 0 & ; \quad 0<x\le\frac{1}{2} \quad \because \quad 0\le\cos\pi x<1 \Rightarrow [\cos\pi x]=0 \\ -1 & ; \quad \frac{1}{2}<x\le 1 \quad \because \quad -1\le\cos\pi x<0 \Rightarrow [\cos\pi x]=-1 \\ -1(3-2x) & ; \quad 1<x\le\frac{3}{2} \quad \because \quad |2x-3|=3-2x \text{ and } [x-2]=-1 \\ -1(2x-3) & ; \quad \frac{3}{2}<x<2 \quad \because \quad |2x-3|=2x-3 \text{ and } [x-2]=-1 \\ 0 & ; \quad x=2 \quad \because \quad [x-2]=0 \end{cases}$$

Checking continuity at $x = 0$:

R.H.L. = $\lim_{x\to 0^+}(0)=0$ and $f(0) = 1$

$\Rightarrow$ $f(x)$ is discontinuous at $x = 0$. [As R.H.L. $\neq f(0)$]

Checking continuity at $x = 1/2$:

L.H.L. = $\lim_{x\to \frac{1}{2}^-} f(x)=0$

R.H.L. = $\lim_{x\to \frac{1}{2}^+} f(x)=-1$

$f(x)$ is discontinuous at $x=\dfrac{1}{2}$. [As L.H.L. $\neq$ R.H.L.]

Checking continuity at $x = 1$:

L.H.L. = $\lim_{x\to 1^-} f(x)=-1$

R.H.L. = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 3) = -1$ and $f(1) = -1$

$f(x)$ is continuous at $x = 1$. [As L.H.L. = R.H.L. = $f(1)$]

Checking continuity at $x = 3/2$:

L.H.L. = $\lim_{x \to \frac{3^-}{2}} (2x - 3) = 0$

R.H.L. = $\lim_{x \to \frac{3^+}{2}} (3 - 2x) = 0$ and $f\left(\frac{3}{2}\right) = 0$

$f(x)$ is continuous at $x = 3/2$. $\left[\text{As L.H.L.} = \text{R.H.L.} = f\left(\frac{3}{2}\right)\right]$

Checking continuity at $x = 2$:

L.H.L. = $\lim_{x \to 2^-} (3 - 2x) = -1$ and $f(2) = 0$

$f(x)$ is discontinuous at $x = 2$. [As L.H.L $\neq f(2)$]

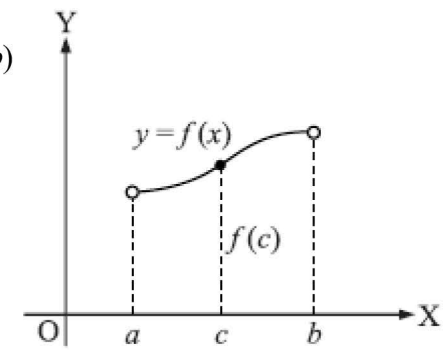
3.3 Continuity in an interval

(i) A function $f(x)$ is said to be continuous in the interval (a, b)

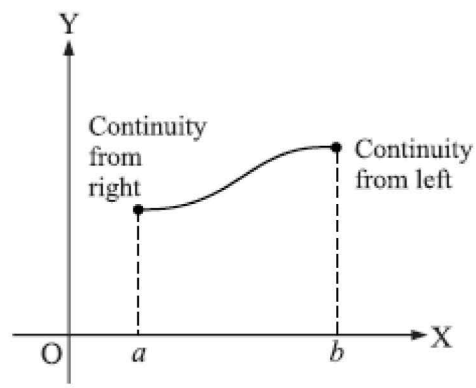
if $f(x)$ is continuous at each and every point $\in (a, b)$

For any $c \in (a, b)$, $f(x)$ is continuous if

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$



(ii) A function $f(x)$ is said to be continuous in the closed inteval $[a, b]$ if it is continuous at every point in the interval (a, b) (see above section) and the continuity at the end points is checked according to the following rule



Continuity at $x = a$

$f(x)$ is continuous at $x = a$ if

If $f(a) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$ = R.H.L.

= a finite quantity (Fig.)

L.H.L. should not be evaluated to check continuity $x = a$

Continuity $x = b$

$f(x)$ is continuous at $x = b$

If $\quad f(b) = \lim\limits_{x \to b^-} f(x) = \lim\limits_{h \to 0} f(b-h)$ = L.H.L. = a finite quantity.

R.H.L. should not be evaluated to check continuity $x = b$.

3.4 Discontinuous Functions

A function f is said to be discontinuous at point a of its domain D if it is not continuous there. The point 'a' is then called a point of discontinuity of the function. The discontinuity may arise due to any of the following situations :

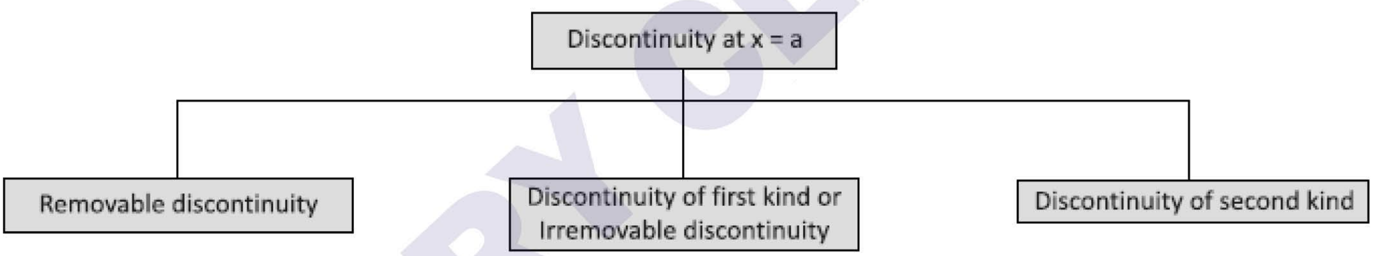
(i) L.H.L. or R.H.L. or both do not exist. at $x = a$
i.e. they either approach to '∞' or $-\infty$ or oscillate between finite or infinite limits.

(ii) L.H.L. as well as R.H.L. exist, but are unequal i.e. L.H.L. $\neq$ R.H.L. at $x = a$.

(iii) L.H.L. as well as R.H.L. both exist and are equal but there values is not equal to $f(a)$
i.e. L.H.L. = R.H.L. $\neq f(a)$ at $x = a$

3.5 Types of Discontinuities

Following are the types of discontinuities



(i) Removal discontinuity :

A function $f(x)$ is said to have a Removable discontinuity at a point $x = a$,

if the limit of $f(x)$ at $x = a$ exists but is not equal to $f(a)$,

i.e., $\quad \lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a+} f(x) \neq f(a)$

$$\underset{(m)}{\lim_{x \to a-} f(x)} = \underset{(m)}{\lim_{x \to a+} f(x)} \neq \underset{(i)}{f(a)}$$

(ii) Discontinuity of the first kind :

The function $f(x)$ is said to have a discontinuity of first kind (or an simple discontinuity) at a point $x = a$
if both L.H.L. and R.H.L. exist but are not equal

The discontinuity of the first kind is also known as jump discontinuity where jump = | R.H.L. – L.H.L |. at $x = a$.

i.e. $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$

Note **:** Discontinuity in this case is non-removable.

(iii) Discontinuity of the second-kind :

A function $y = f(x)$ is said to have a discontinuity of second kind at a point $x = a$ if either or both of the limits

$\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ does not exist i.e.,

if either or both of the limit

$\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ is infinite or as oscillatary

The discontinuity of second kind is also known as essential discontinity.

***Illustrating the Concepts* :**

(i) *If* $f(x) = \frac{1}{1+e^{1/x}}, x \neq 0$, *discuss the continuity of* $f(x)$ *at* $x = 0$.

At $x = 0$

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

and $$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{1}{1+e^{+1/h}} = \frac{1}{1+e^{+\infty}} = \frac{1}{\infty} = 0$$

Hence L. H.L. $\neq$ R.H.L.

$\Rightarrow$ $f(x)$ has a jump discontinuity at $x = 0$.

(ii) *Discuss the continuity of the function* $f(x) = \sin(\log_e |x|)$ *at* $x = 0$.

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \sin(\log_e |0-h|)$$

$$= \lim_{h \to 0} \sin(\log_e h) = \sin(\log_e 0) = \sin(-\infty) = -\sin\infty$$

= oscillating between – 1 and 1.

$$\text{R.H.L.} = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} \sin\left(\log_e |0+h|\right)$$

$$= \lim_{h \to 0} \sin\left(\log_e h\right) = \sin(\log_e 0) = \sin(-\infty) = -\sin\infty$$

= oscilating between – 1 and 1.

Therefore L.H.L. and R.H.L. are undefined.

Hence $f(x)$ has a essential discontinuity.

(iii) *A function $f(x)$ satisfies the following property :*

$$f(x+y) = f(x)\,f(y)$$

Show that the function is continuous for all values of x if it is continuous at $x = 1$.

As the function is continuous at $x = 1$, we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad \lim_{h \to 0} f(1-h) = \lim_{h \to 0} f(1+h) = f(1) \qquad [\text{using } f(x+y) = f(x)\,f(y)]$$

we get,

$$\Rightarrow \quad \lim_{h \to 0} f(1)\,f(-h) = \lim_{h \to 0} f(1)\,f(h) = f(1)$$

$$\Rightarrow \quad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 1 \qquad \text{. . . (i)}$$

Now consider some arbitrary point $x = a$.

$$\text{Left hand limit} = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a)\,f(-h)$$

$$= f(a) \lim_{h \to 0} f(-h) = f(a) \qquad [\text{using (i)}]$$

$$\text{Right hand limit} = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a)\,f(h)$$

$$= f(a) \lim_{h \to 0} f(h) = f(a) \qquad [\text{using (i)}]$$

Hence at any arbitrary point $(x = a)$,

L.H.L. = R.H.L. = $f(a)$

$\Rightarrow$ function is continuous for all values of x.

Illustration - 37

If $g(x) = f(f(x))$ *where* $f(x)=\begin{cases}1+x & ; \quad 0\le x\le 2\\ 3-x & ; \quad 2<x\le 3\end{cases}$ *then the number of point of discontinuity of* $g(x)$ *in* $[0, 3]$ *is* :

(A) 2 **(B)** 3 **(C)** 4 **(D)** 0

SOLUTION : (A)

$$g(x)=f(f(x))=\begin{cases}f(1+x) & ; \quad 0\le x\le 2\\ f(3-x) & ; \quad 2<x\le 3\end{cases}$$

$$=\begin{cases}f(1+x) & ; \quad 0\le x\le 1\\ f(1+x) & ; \quad 1<x\le 2\\ f(3-x) & ; \quad 2<x\le 3\end{cases}$$

now $x \in [0, 1] \Rightarrow (1+x) \in [1, 2]$

$x \in (1, 2] \Rightarrow (1+x) \in (2, 3]$

$x \in (2, 3] \Rightarrow (3-3) \in [0, 1)$

Hence

$$g(x)=\begin{cases}f(1+x) & \text{for} \quad 0\le x\le 1 \Rightarrow 1\le x+1\le 2\\ f(1+x) & \text{for} \quad 1<x\le 2 \Rightarrow 2<x+1\le 3\\ f(3-x) & \text{for} \quad 2<x\le 3 \Rightarrow 0\le 3-x<1\end{cases}$$

Now if $(1+x) \in [1, 2]$ then

$f(1+x) = 1 + (1+x) = 2 + x$ **. . . (i)**

[from the original definition of $f(x)$]

Similarly if $(1+x) \in (2, 3)$ then

$f(1+x) = 3 - (1+x) = 2 - x$ **. . . (ii)**

If $(3-x) \in (0, 1)$ then

$f(3-x) = 1 + (3-x) = 4 - x$ **. . . (iii)**

Using (i), (ii) and (iii), we get :

$$g(x)=\begin{cases}2+x & ; \quad 0\le x\le 1\\ 2-x & ; \quad 1<x\le 2\\ 4-x & ; \quad 2<x\le 3\end{cases}$$

Now we will check the continuity of $g(x)$ at $x = 1, 2$.

At $x = 1$

L.H.L. $= \lim\limits_{x\to 1^-} g(x) = \lim\limits_{x\to 1^-}(2+x)=3$

R.H.L. $= \lim\limits_{x\to 1^+} g(x) = \lim\limits_{x\to 1^+}(2-x)=1$

[As L.H.L. $\ne$ R.H.L., $g(x)$ is discontinuous at $x = 1$]

At $x = 2$

L.H.L. $= \lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-}(2-x)=0$

R.H.L. $= \lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+}(4-x)=2$

[As L.H.L. $\ne$ R.H.L., $g(x)$ is discontinuous at $x = 2$]

Illustration - 38

The natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ *where the function f satisfies the relation* $f(x+y) = f(x)\,f(y)$ *for all natural numbers x, y and further* $f(1) = 2$ *is :*

(A) 2 **(B)** 3 **(C)** 1 **(D)** *None of these*

SOLUTION : (B)

Since the function f satisfies the relation

$$f(x+y) = f(x)\,f(y)$$

It must be an exponential function.

Let the base of this exponential function be a.

Thus $f(x) = a^x$

It is given that $f(1) = 2$. So we can make

$$f(1) = a^1 = 2 \Rightarrow \quad a = 2$$

Hence, the function is $f(x) = 2^x$ **. . . (i)**

[Alternatively, we have]

$$f(x) = f(x-1+1) = f(x-1)\,f(1)$$

$$= f(x-2+1)\,f(1)$$

$$= f(x-2)\,[f(1)]^2 = \ldots = [f(1)]^x = 2^x$$

where x is an integer

Using equation (i), the given expression reduces to :

$$\sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^a \cdot 2^k = 16(2^n - 1)$$

$$\Rightarrow \quad 2^a \sum_{k=1}^{n} 2^k = 16(2^n - 1)$$

$$\Rightarrow 2^a (2 + 4 + 8 + 16 + \ldots + 2^n) = 16(2^n - 1)$$

$$\Rightarrow \quad 2^a \left[\frac{2(2^n - 1)}{2 - 1}\right] = 16(2^n - 1)$$

$$\Rightarrow \quad 2^{a+1} = 16 \Rightarrow \quad 2a + 1 = 2^4$$

$$\Rightarrow \quad a + 1 = 4 \Rightarrow \quad a = 3$$

DIFFERENTIABILITY

4.1 Definition

The derivative of function $y = f(x)$ is defined as the instantaneous rate of change of y {or $f(x)$} with respect to the change in the independent variable x.

$$\text{Derivative} = \lim_{h \to 0} \frac{\text{change in } y}{\text{change in } x}$$

As x changes from x to $x + h$, y changes from $f(x)$ to $f(x + h)$. Hence

$$\text{Derivative} = \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

4.2 Existence of derivative (Differentiability) at a point

We have already defined deriavative of y with respect to x as the instantaneous rate of change of y with respect to x.

Consider an arbitary point $x = a$.

(a) If x changes from a to $a + h$, derivative at $x = a$ is :

Right Hand Derivative = $R\ f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

(b) If x changes from a to $a - h$, derivative at $x = a$ is :

Left Hand Derivative = $L\,f'(a) = \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

We say that derivative at $x = a$ exists or the function is differentiable at $x = a$ if both the left hand derivative and the right hand derivative are finite and equal.

$\Rightarrow$ $R\,f'(a) = L\,f'(a)$ is the condition for differentiability at $x = a$.

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

4.3 Another Expression for $f'(a)$

We can also find derivative of $f(x)$ at $x = a$ with the use of the following formula :

$$f'(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}$$

4.4 Geometrical Meaning of Derivative

4.4.1 Geometrical meaning of Right hand derivative

Let $P\,(a, f(a))$ and $Q\,(a + h, f(a + h))$ be two points very near to each other on the curve $y = f(x)$.

Using slope of a line formula, we get

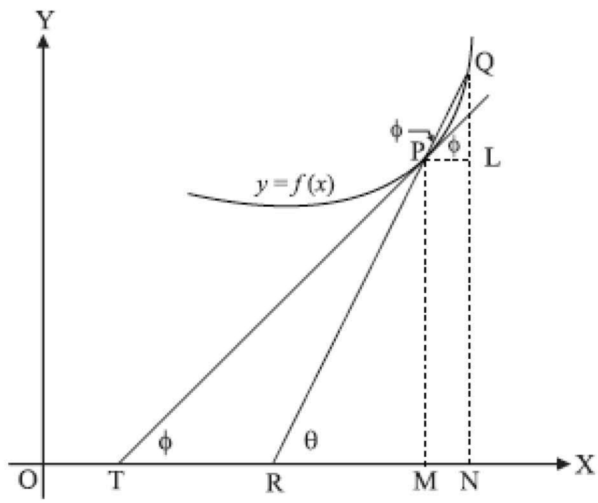
$$\text{Slope of } PQ = \frac{f(a+h) - f(a)}{(a+h) - a}$$

Now apply $\lim\limits_{h \to 0}$ on both sides to get :

$$\lim_{h \to 0} (\text{slope of chord } PQ) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Right hand derivative

$$= R\left[f'(a)\right] = \lim_{h \to 0} (\text{slope of chord } PQ)$$



As $h \to 0$, $Q \to P$ on curve, $a + h \to a$ on x-axis and $f(a+h) \to f(a)$ on y-axis.

When h is infinitely small, chord PQ almost becomes tangent drawn at P towards right i.e.

$$R'[f(a)] = \lim_{h \to 0} (\text{slope of chord } PQ)$$

$$= \text{slope of tangent drawn at P towards right.}$$

Hence geometrical significence of right hand derivative is that it represents slope of tangent drawn at P towards right.

4.4.2 Geometrical meaning of Left hand derivative

Let $P(a, f(a))$ and $Q(a-h, f(a-h))$ be two points very near to each other on the curve $y = f(x)$.

Using slope of a line formula, we get : Slope of $PQ = \dfrac{f(a-h) - f(a)}{(a-h) - a}$

Now apply $\lim\limits_{h \to 0}$ on both sides to get : $\lim\limits_{h \to 0} (\text{slope of chord } PQ) = \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

Left hand derivative $= L[f'(a)] = \lim\limits_{h \to 0} (\text{slope of chord } PQ)$

As $h \to 0$, $Q \to P$ on curve, $a - h \to a$ on x-axis and $f(a-h) \to f(a)$ on y-axis. When h is infinitely small, chord PQ almost becomes tangent drawn at P towards left i.e.

$$L'[f(a)] = \lim_{h \to 0} (\text{slope of chord } PQ)$$

$$= \text{slope of tangent drawn at P towards left.}$$

Hence geometrical significence of left hand derivative is that it represents slope of tangent drawn at P towards left.

4.4.3 Geometrical meaning of existence of derivative

We know derivative exists at $x = a$, if $L[f'(a)] = R[f'(a)]$

$\Rightarrow$ Slope of tangent drawn at P towards left = slope of tangent drawn at P towards right

$\Rightarrow$ Same tangent line towards left and right

$\Rightarrow$ Smooth curve around $x = a$

Hence if $f(x)$ is differentiable or derivative at $x = a$ exists, then at $x = a$ we can drawn only one tangent towards left and right.

i.e. curve would be smooth in the neighbourhood of a.

4.5 Existence of derivative (Differentiability) on an interval

Let $y = f(x)$ is a function which is defiend in the closed interval $[a, b]$.

(a) If $f(x)$ is a differentiable at every point on the open interval (a, b), then $f(x)$ is said to be differentiable on (a, b).

(b) If $f(x)$ is differentiable on (a, b) and $f'(a^+)$ and $f'(b^-)$ exists finitely, then $f(x)$ is said to be differentiable on closed interval $[a, b]$.

4.6 Results

(a) If $f(x)$ is defferentiable at $x = a$, the it must be continuous at $x = a$ or if $f(x)$ is differentiable on the interval (a, b), then it must be continuus for all x lying in this interval.

(b) The converse of above result is not true i.e. if function is continuous at $x = a$, then it may or may not be differentiable at $x = a$. OR if function is continuous on the interval (a, b) then it may or may not differentiable for all x in that interval.

(c) If $Rf'(a)$ and $Lf'(a)$ both exist finitely (both may or may not be equal) then $f(x)$ is continuous at $x = a$.

(d) If a function is differentiable, its graph must be smooth i.e. there should be no break or corner.

Illustrating the Concepts :

(i) *Discuss the differentiability $f(x)$ at $x = -1$, if* $f(x) = \begin{cases} 1 - x^2 & ; \quad x \le -1 \\ 2x + 2 & ; \quad x > -1 \end{cases}$

$$f(-1) = 1 - (-1)^2 = 0$$

Right hand derivative at $x = -1$ is

$$Rf'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{2(-1+h) - 2 - 0}{h} = \lim_{h \to 0} \frac{2h}{h} = 2.$$

$$Lf'(-1) = \lim_{h \to 0} \frac{f(-1-h) - f(-1)}{-h}$$

$$= \lim_{h \to 0} \frac{1 - (-1-h)^2 - 0}{-h} = \lim_{h \to 0} \frac{-h^2 - 2h}{-h} = \lim_{h \to 0} (h+2) = 2$$

Since $Lf'(-1) = Rf'(-1) = 2$.

$\Rightarrow$ The function is differentiable at $x = -1$.

(ii) *Show that the functin $f(x) = |x^2 - 4|$ is not differentiable at $x = 2$.*

$$f(x) = \begin{cases} x^2 - 4 & ; \quad x \le -2 \\ 4 - x^2 & ; \quad -2 < x < 2 \\ x^2 - 4 & ; \quad x \ge 2 \end{cases}$$

$\Rightarrow \quad f(2) = 2^2 - 4 = 0$

$$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \frac{4 - (2-h)^2 - 0}{-h}$$

$$= \lim_{h \to 0} \frac{4h - h^2}{-h} = \lim_{h \to 0} (h - 4) = -4.$$

$$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{[(2+h)^2 - 4] - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} (h + 4) = 4$$

$\Rightarrow \quad Lf'(2) \ne Rf'(2).$

Hence $f(x)$ is not differentiable at $x = 2$.

(iii) *Show that $f(x) = x\,|x|$ is differentiable $x = 0$.*

$$f(x) = \begin{cases} -x^2 & ; \quad x \le 0 \\ x^2 & ; \quad x > 0 \end{cases}$$

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(-h)^2 - 0}{-h} = \lim_{h \to 0} h = 0$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 - 0}{h} = 0 \quad \Rightarrow \quad Lf'(0) = Rf'(0).$$

Hence $f(x)$ is differentiable at $x = 0$.

(iv) *Prove the following theorem :*

"If a function $y = f(x)$ is differentiable at a point then it must be continuous at that point".

Let the function be differentiable at $x = a$.

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

and $$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

are finite numbers which are equal.

$$\text{L.H.L.} = \lim_{h \to 0} f(a-h)$$

$$= \lim_{h \to 0} [f(a-h) - f(a)] + f(a)$$

$$= \lim_{h \to 0} (-h)\left[\lim_{h \to 0} \frac{f(a-h) - (a)}{-h}\right] + f(a)$$

$$= 0 \times [Lf'(a)] + f(a) = f(a)$$

$$\text{R.H.L.} = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} [f(a+h) - f(a)] + f(a)$$

$$= \lim_{h \to 0} h\left[\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\right] + f(a)$$

$$= 0 \times [R f'(a)] + f(a) = f(a)$$

Hence the function is continuous at $x = a$.

Note: The converse of this theorem is not always true. If a function is continuous at a point, it may or may not be differentiable at that point.

Illustration - 39 *At $x = 0$ the given function*

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases} \quad \text{is :}$$

(A) *Discontinuous* **(B)** *Differentiable* **(C)** *Non-differentiable* **(D)** *None of these*

SOLUTION : (B)

Let us check the differentiability first.

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = \lim_{h \to 0} h \times \lim_{h \to 0} \sin\frac{1}{h}$$

$= 0 \times$ (number between -1 and $+1$) $= 0$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = \lim_{h \to 0} h \times \lim_{h \to 0} \sin\frac{1}{h}$$

$= 0 \times$ (number between -1 and $+1$) $= 0$

Hence $L f'(0) = R f'(0) = 0$.

$\Rightarrow$ Function is differentiable at $x = 0$.

$\Rightarrow$ It must be continuous also at the same point.

Illustration - 40 *At $x = 0$ the given function*

$$f(x) = \begin{cases} x \sin(\log x^2) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases} \quad is:$$

(A) *Discontinuous* **(B)** *Differentiable* **(C)** *Non-differentiable* **(D)** *None of these*

SOLUTION : (C)

$$\text{LHL} = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} (-h) \sin \log (-h)^2$$

$$= -\lim_{h \to 0} h \sin \log h^2$$

As $h \to 0$, $\log h^2 \to -\infty$.

Hence $\sin \log h^2$ oscillates between – 1 and + 1.

$$\Rightarrow LHL = -\lim_{h \to 0} (h) \times \lim_{h \to 0} (\sin \log h^2)$$

$$= -0 \times (\text{number between } -1 \text{ and } +1) = 0$$

$$\text{R.H.L.} = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} h \sin \log h^2 = \lim_{h \to 0} h \lim_{h \to 0} \sin \log h^2$$

$$= 0 \times (\text{oscillating between } -1 \text{ and } +1) = 0$$

$f(0) = 0$ (Given)

$\Rightarrow$ L.H.L.= R.H.L. $= f(0)$

Hence $f(x)$ is continuous at $x = 0$.

Test for Differentiability :

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \sin \log (-h)^2 - 0}{-h}$$

$$= \lim_{h \to 0} \sin (\log h^2)$$

As the expression oscillates between – 1 and + 1, the limit does not exist.

$\Rightarrow$ Left hand derivative is not defined.

Hence the function is not differentiable at $x = 0$.

Note : As LHD is undefined there is no need to check RHD for differentiability as for differentiability both LHD and RHD should be defined and equal.

Illustration - 41 *For the given function*

$$f(x)=\begin{cases}\dfrac{x^2}{2} & ; \quad 0\le x<1\\ 2x^2-3x+\dfrac{3}{2} & ; \quad 1\le x\le 2\end{cases}$$ *which of the following is (are) correct :*

(A) $f(x)$ *is continuous* $\forall x\in[0,2]$ **(B)** $f'(x)$ *is continuous* $\forall x\in[0,2]$

(C) $f''(x)$ *is discontinuous at* $x=1$ **(D)** $f''(x)$ *is continuous* $\forall x\in[0,2]$

SOLUTION : (ABC)

Continuity of $f(x)$

For $x\neq 1$, $f(x)$ is a polynomial and hence is continuous.

At $x=1$,

$$\text{LHL} = \lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{x^2}{2}=\frac{1}{2}$$

$$\text{RHL}=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(2x^2-3x+\frac{3}{2}\right)$$

$$=2-3+\frac{3}{2}=\frac{1}{2}$$

$$f(1)=2(1)^2-3(1)+\frac{3}{2}=\frac{1}{2}$$

$\Rightarrow$ L.H.L. = R.H.L. = $f(1)$

Therefore, $f(x)$ is continuous at $x=1$.

Continuity of $f'(x)$

Let $g(x)=f'(x)$

$$\Rightarrow \quad g(x)=\begin{cases}x & ; \quad 0\le x<1\\ 4x-3 & ; \quad 1\le x\le 2\end{cases}$$

For $x\neq 1$, $g(x)$ is linear polynomial and hence continuous.

At $x=1$,

$$\text{LHL}=\lim_{x\to 1^-} g(x)=\lim_{x\to 1^-} x=1$$

$$\text{RHL}=\lim_{x\to 1^+} g(x)=\lim_{x\to 1^+}(4x-3)=1$$

$g(1)=4-3=1$

$\Rightarrow$ LHL = RHL = $g(1)$

$\therefore$ $g(x)=f'(x)$ is continuous at $x=1$.

Continuity of $f''(x)$

Let $$h(x)=f''(x)=\begin{cases}1 & ; \quad 0\le x<1\\ 4 & ; \quad 1\le x\le 2\end{cases}$$

For $x\neq 1$, $h(x)$ is continuous because it is a constant function.

At $x=1$,

$$\text{LHL}=\lim_{x\to 1^-} h(x)=1$$

$$\text{RHL}=\lim_{x\to 1^+} h(x)=4$$

Thus LHL $\neq$ RHL

$\therefore$ $h(x)$ is discontinuous at $x=1$

Hence $f(x)$ and $f'(x)$ are continuous on [0, 2] but $f''(x)$ is discontinuous at $x=1$.

Note **:** Continuity of $f'(x)$ is same as differentiability of $f(x)$.

Illustration - 42 *If $f(x)$ and $g(x)$ are differentiable at $x = a$ then the value of*

$$\lim_{x \to a} \frac{f(x)\,g(a) - g(x)\,f(a)}{x - a} \text{ is :}$$

(A) $f(a)\,g(a) - f'(a)\,g'(a)$ **(B)** $f'(a)\,g(a) - g'(a)\,f(a)$

(C) $f(a)\,g'(a) - f'(a)\,g(a)$ **(D)** *None of these*

SOLUTION : (B)

$$\lim_{x \to a} \frac{f(x)\,g(a) - g(x)\,f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{f(x)\,g(a) - f(a)\,g(a) + f(a)\,g(a) - g(x)\,f(a)}{x - a}$$

$$= \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a}\right] g(a) - \lim_{x \to a}\left[\frac{g(x) - g(a)}{x - a}\right] f(a)$$

$$= f'(a)\,g(a) - g'(a)\,f(a)$$

Illustration - 43 *Let $f(x)$ be defined in the interval $[-2, 2]$ such that*

$$f(x) = \begin{cases} -1 & ; \quad -2 \le x \le 0 \\ x - 1 & ; \quad 0 < x \le 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$. The number of point where $g(x)$ is not differentiable in $(-2, 2)$ is :

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

SOLUTION : (B)

Consider $f(|x|)$

The given interval is $-2 \le x \le 2$

Replace x by $|x|$ to get :

$$-2 \le |x| \le 2 \quad \Rightarrow \quad 0 \le |x| \le 2$$

Hence $f(|x|)$ can be obtained by substituting $|x|$ in place of x in $x - 1$

[see definition of $f(x)$].

$$\Rightarrow \quad f(|x|) = |x| - 1; \quad -2 \le x \le 2 \quad \ldots \textbf{(i)}$$

Consider $|f(x)|$

$$\text{Now } |f(x)| = \begin{cases} |-1| & ; \quad -2 \le x \le 0 \\ |x - 1| & ; \quad 0 < x \le 2 \end{cases}$$

$$\Rightarrow \quad |f(x)| = \begin{cases} 1 & ; \quad -2 \le x \le 0 \\ |x - 1| & ; \quad 0 < x \le 2 \end{cases} \quad \ldots \textbf{(ii)}$$

Adding (i) and (ii) we get :

$$f(|x|) + |f(x)| = \begin{cases} |x| - 1 + 1 & ; \quad -2 \le x \le 0 \\ |x| - 1 + |x - 1| & ; \quad 0 < x \le 2 \end{cases}$$

$$\Rightarrow g(x)=\begin{cases} |x| & ; \quad -2 \le x \le 0 \\ |x|-1+|x-1| & ; \quad 0 < x \le 2 \end{cases}$$

On further simplification,

$$g(x)=\begin{cases} -x & ; \quad -2 \le x \le 0 \\ x-1+1-x & ; \quad 0 < x < 1 \\ x-1+x-1 & ; \quad 1 \le x \le 2 \end{cases}$$

$$g(x)=\begin{cases} -x & ; \quad -2 \le x \le 0 \\ 0 & ; \quad 0 < x < 1 \\ 2x-2 & ; \quad 1 \le x \le 2 \end{cases}$$

For $x \neq 0$ and $x \neq 1$, $g(x)$ is a differentiable function because it is a linear polynomial.

At $x = 0$

$$Lg'(0)=\lim_{h \to 0} \frac{g(0-h)-g(0)}{-h}$$

$$=\lim_{h \to 0} \frac{-(-h)-0}{-h}=-1$$

$$Rg'(0)=Rg'(0)=\lim_{h \to 0} \frac{g(0+h)-g(0)}{h}$$

$$=\lim_{h \to 0} \frac{0-0}{h}=0$$

$\Rightarrow Lg'(0) \neq Rg'(0)$.

Therefore $g(x)$ is not differentiable at $x = 0$.

At $x = 1$

$$Lg'(1)=\lim_{h \to 0} \frac{g(1-h)-g(1)}{-h}$$

$$=\lim_{h \to 0} \frac{0-0}{-h}=0$$

$$Rg'(1)=\lim_{h \to 0} \frac{g(1+h)-g(1)}{h}$$

$$=\lim_{h \to 0} \frac{2(1+h)-2-0}{h}=2$$

$\Rightarrow Lg'(1) \neq Rg'(1)$.

Therefore $g(x)$ in not differential at $x = 1$.

Hence $g(x)$ is not differentiable at $x = 0, 1$ in $(-2, 2)$.