SOLUTIONS

BASIC MATHEMATICS USED IN PHYSICS AND VECTORS

1. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}...$ using G.P.

$a + ar + ar^2... ar^n$ where $a = 1$ & $r = \frac{1}{2}$

$sum = \frac{a}{1-r} = \frac{1}{1-1/2} = 2$

2. $\int \frac{dx}{2x-3} = \frac{\ln(2x-3)}{2} + c$

3. by binomial expainsion $\left(1+\frac{h}{R}\right)^{-2} = \left(1-\frac{2h}{R}\right)$

if $h << R$

4. $2x^2 - 3x + 5 = 0 \Rightarrow x^2 - \frac{3}{2}x + \frac{5}{2} = 0$

sum or product of root for $ax^2 + bx + c = 0$ is $\frac{-b}{a}$

and product of root is $\frac{c}{a}$

5. $(0.996)^{1/4} = (1 - 0.004)^{1/4}$

also for $(1 + x)^n$ and $|x| <<< 1$

$(1 + x)^n = (1 + nx)$

$(1 - 0.004)^{1/4} = \left(1-\frac{0.004}{4}\right) = 0.999$

6. $\int \sin 4x\, dx = -\frac{\cos 4x}{4} + c$

7. $\phi = 2t^2 - 3t + 4$

$e = -\frac{d\phi}{dt} = -4t + 3$ at $t = 2$

$e = -8 + 3 = -5$ units

8. $\sin 480° = \sin(540 - 60°) = \sin(60°) = \frac{\sqrt{3}}{2}$

9. $y = x \sin x \Rightarrow \frac{dy}{dx} = x(\cos x) + 1 \sin x$

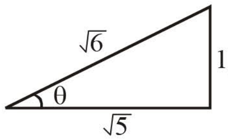
$\frac{dy}{dx} = x \cos x + \sin x$

10. $\int \cos^2\theta d\theta$

$\cos^2\theta - \sin^2\theta = \cos 2\theta \Rightarrow 2\cos^2\theta - 1 = \cos 2\theta$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\int \cos^2\theta\, d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + c$

11. 

$\Rightarrow \cos\theta = \frac{\sqrt{5}}{\sqrt{6}}$

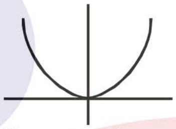
12. $y = \sin(4x - 3) \Rightarrow \frac{dy}{dx} = 4\cos(4x-3)$

13. $3x - 2y + 4 = 0 \Rightarrow 2y = 3x + 4$

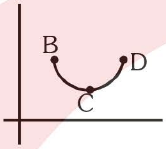
$y = \frac{3}{2}x + \frac{4}{2} \Rightarrow y = +\frac{3}{2}x + 2$

slope $= +m = \frac{3}{2}$ & $c = +2$

14. $y = x^2$ represent



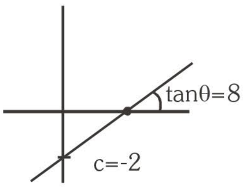
15. Continously increasing slope.



16. $v = 4t^2 - 2t$

$a = \frac{dV}{dt} = 8t - 2$

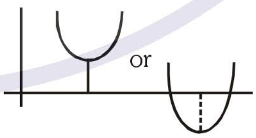
a = represent straight line



17. for minima

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

so graph should be



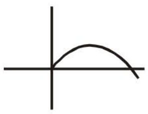
18. $y = 2x - 4x^2$

$y = -4x^2 + 2x$

$y = x(-4x + 2)$

$y = 0$ at $x = 0$

$x = 1/2$



also $-4x^2$ represent downward parabola

20. $\because$ $f = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$ $\therefore$ $f^2 \propto \frac{1}{\ell}$

22. $v_{av} = \dfrac{\int v\,dt}{\int dt} = \dfrac{\int_0^1 (2t+3)dt}{\int_0^1 dt} = \dfrac{(t^2+3t)_0^1}{(t)_0^1} = 4 m/s$

23. As $K = \dfrac{1}{2}mv^2 \geq 0$ so $9 - x^2 \geq 0 \Rightarrow -3 \leq x \leq 3$

24. $3\cos\theta + 4\sin\theta = 5\left[\dfrac{3}{5}\cos\theta + \dfrac{4}{5}\sin\theta\right]$

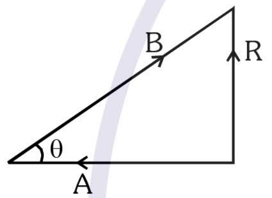
$= 5[\sin\alpha\cos\theta + \cos\alpha\sin\theta] = 5\sin(\theta+\alpha)$

Therefore A = 5 and $\alpha = \sin^{-1}\left(\dfrac{3}{5}\right)$

26. $F_1 - F_2 \leq |\vec{F}_1 + \vec{F}_2| \leq F_1 + F_2$

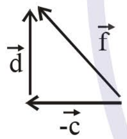
F_3 must lie between $F_1 - F_2 \leq F_3 \leq F_1 + F_2$ to produce zero resultant.

29.



$\sin\theta = \dfrac{R}{B} = \dfrac{B/\sqrt{2}}{B} = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = 45°$

30.



31. $\overrightarrow{\Delta r} = \overrightarrow{OB} - \overrightarrow{OA}$

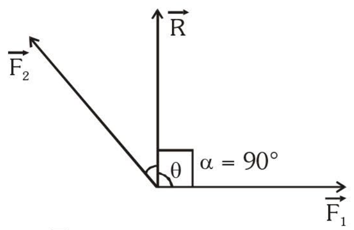
$|\overrightarrow{\Delta r}| = \sqrt{\ell^2 + \ell^2 - 2\ell^2\cos\theta} = \sqrt{2\ell^2(1-\cos\theta)}$

$\sqrt{2\ell^2 \times 2\sin^2\theta/2} = 2\ell\sin\left(\dfrac{\theta}{2}\right)$

32. $F_1 + F_2 = 16$ ...(i)

$8 = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$

$\Rightarrow F_1^2 + F_2^2 + 2F_1F_2\cos\theta = 64$ ...(ii)

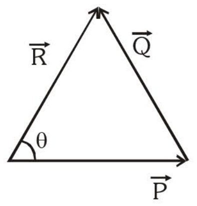


$\tan\alpha = \dfrac{F_2\sin\theta}{F_1 + F_2\cos\theta} = \tan 90° = \infty$

$F_2\cos\theta = -F_1$ ...(iii)

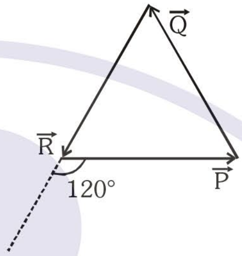
Solving (i), (ii) and (iii) $\Rightarrow F_1 = 6N, F_2 = 10N$

33. P = Q = R and $\vec{P} + \vec{Q} = \vec{R}$



$\theta_1 = 60°$

If P = Q = R and $\vec{P} + \vec{Q} + \vec{R} = 0$



$\theta_2 = 120°$

34. If θ and θ' are the angles made by resultant velocities of first and second ball respectively from the x-axis, then

$\tan\theta = \dfrac{v_y}{v_x} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$ or $\theta = 60°$

and $\tan\theta' = \dfrac{v'_y}{v'_x} = \dfrac{2}{2} = 1$ or $\theta' = 45°$

Angle between the paths of the balls

$= \theta - \theta' = 60° - 45° = 15°$

35. Let θ is the angle between the vectors

$\therefore A^2 = A^2 + A^2 + 2AA\cos\theta$

which gives $\cos\theta = -\dfrac{1}{2}$ or $\theta = 120°$

36. The second's needle gets rotated by 90° in 15 seconds,

so $\Delta v = \sqrt{v^2 + v^2 - 2vv\cos 90°}$

$= \sqrt{2}v = \sqrt{2}\omega r = \sqrt{2} \times \dfrac{2\pi}{60} \times 1 = \dfrac{\pi\sqrt{2}}{30}$ cm/s

37. $\tan\theta/2 = \dfrac{B\sin\theta}{A + B\cos\theta}$ $\therefore A = B$

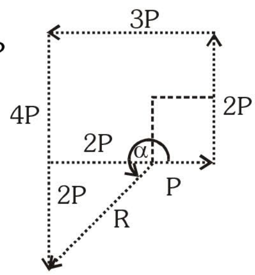
39. The forces are drawn along the sides of a square as shown in the figure. It is clear from the figure that

$R = \sqrt{(2P)^2 + (2P)^2} = 2\sqrt{2}P$

and it makes angle

$\alpha = 180° + 45°$

$= 225°$ with x-axis



40. $|\vec{V}_1 + \vec{V}_2| = |\vec{V}_1 - \vec{V}_2|$

or $V_1^2 + V_2^2 + 2V_1V_2\cos\theta = V_1^2 + V_2^2 - 2V_1V_2\cos\theta$

or $\cos\theta = 0 \quad \therefore \theta = 90°$

41. If $\vec{A}$ is the required vector, then

$\bar{A} + (i - 5\hat{j} + 2\hat{k}) + (3\hat{i} + 6\hat{j} - 7\hat{k}) = i \text{ or } \hat{j} \text{ or } \hat{k}$

$\therefore \vec{A} = -3\hat{i} - \hat{j} + 5\hat{k}$

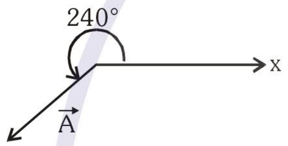
42. For parallel vectors

$$\frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3}$$

43. The required vector is $B\hat{A} = \frac{B\vec{A}}{A}$

$$= \sqrt{7^2 + 24^2} \times \frac{(3\hat{i} + 4\hat{j})}{\sqrt{3^2 + 4^2}} = 15i + 20j$$

44.



$A_x = A\cos 240° = -\frac{A}{2}$

and $A_y = A\sin 240° = -\sqrt{3}\,\frac{A}{2}$

45. Let $\vec{b} = (\hat{i} + \hat{j})$

The component of $\vec{a}$ along $\vec{b}$

$$a\cos\theta\ \hat{b} = \left(\frac{\vec{a}.\vec{b}}{b}\right)\hat{b}$$

$$= \frac{(2\hat{i} + 3\hat{j})\ (\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}} \frac{(\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}}$$

$$= \frac{2 \times 1 + 3 \times 1}{\sqrt{2}} \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = \frac{5}{2}(\hat{i} + \hat{j})$$

46. Angle with y-axis $\Rightarrow \tan\theta = \frac{x - \text{comp}}{y - \text{comp}} = \frac{2}{3}$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

47. $\vec{A} = \hat{i} + \hat{j} + \sqrt{2}\hat{k}$

$$\cos\alpha = \frac{Ax}{A}, \cos\beta = \frac{Ay}{A}, \cos\gamma = \frac{Az}{A}$$

$$\cos\alpha = \frac{1}{2}, \cos\beta = \frac{1}{2}, \cos\gamma = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

48. If three unit vectors are along the same direction, then their resultant will be 3. The resultant may be zero when each one has angle 120° from other.

49. If A, B and C are the magnitudes of three vectors, then for their resultant to be zero.

$A - B \le C \le A + B$

50. The dot product of $5\hat{k}$ with $4\hat{i} + 3\hat{j}, 6\hat{i}$ and $3\hat{i} + 4\hat{j}$ is zero, so these are perpendicular vectors to $5\hat{k}$

51. Let eastern line be taken as x-axis, northern as y-axis and vertical upward as z-axis. Let the velocity v makes angle α, β and γ with x, y and z-axis respectively, then $\alpha = 60°$, $\gamma = 60°$

we have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or $\cos^2 60 + \cos^2\beta + \cos^2 60 = 1$ or $\cos\beta = \frac{1}{\sqrt{2}}$

$\therefore \vec{v} = v\cos\alpha i + v\cos\beta\hat{j} + v\cos\gamma\hat{k}$

$$= 20\left[\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right] = 10\hat{i} + 10\sqrt{2}\hat{j} + 10\hat{k}$$

52. $(2\hat{i} + 3\hat{j} + 8\hat{k}).(4\hat{j} - 4\hat{i} + \alpha\hat{k}) = 0$

$-8 + 12 + 8\alpha = 0 \Rightarrow 8\alpha = -4$

$$\alpha = -1/2$$

53. $\vec{P}.\vec{Q} = 0 \Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0$

$a^2 - 2a - 3 = 0 \Rightarrow (a - 3)(a + 1) = 0 \Rightarrow a = 3, -1$

54. $\vec{A} \cdot \vec{B} = 0$

$$\cos\omega t\cos\frac{\omega t}{2} - \sin\omega t\sin\frac{\omega t}{2} = 0$$

$$\cos\left(\omega t + \frac{\omega t}{2}\right) = 0 \Rightarrow \cos\frac{3\omega t}{2} = 0$$

$$\Rightarrow \frac{3\omega t}{2} = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{3\omega}$$

55. $(\vec{A} + \vec{B}).(\vec{A} - \vec{B}) = 0$

$\Rightarrow A^2 - \vec{A}.\vec{B} + \vec{B}.\vec{A} - B^2 = 0$

$\Rightarrow A = B\ (\because \vec{A}.\vec{B} = \vec{B}.\vec{A})$

56. $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{1+1}{\sqrt{3}\cdot\sqrt{2}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

57. $\vec{A} = a\hat{k}, \quad \vec{B} = b\hat{j}$

$\vec{A}\times\vec{B} = ab(\hat{k}\times\hat{j})$

= ab (–i) = ab (along west)

58. $\vec{A}\times\vec{B} = \vec{0} \Rightarrow \vec{A} \mid\mid \vec{B}$

$\vec{B}\times\vec{C} = \vec{0} \Rightarrow \vec{B} \mid\mid \vec{C}$

thus $\vec{A}$ and $\vec{C}$ may be parallel

59. $(\vec{A}\times\vec{B})$ will be perpendicular to the plane of $\vec{A}$ and $\vec{B}$

61. $|\vec{A}\times\vec{B}| = \sqrt{3}\vec{A}.\vec{B} \Rightarrow AB\sin\theta = \sqrt{3}\,AB\cos\theta$

$\Rightarrow \quad \tan\theta = \sqrt{3} \Rightarrow \theta = 60°$

R = $\sqrt{A^2+B^2-2AB\cos 60°} = (A^2 + B^2 - AB)^{1/2}$

62. Area of parallelogram = $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$\vec{d}_1$ and $\vec{d}_2$ are diagonals.

63. $\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \tan 90° = \infty$

thus, A + B cos θ = 0 ⇒ cosθ = –A/B

64. $\vec{R} = (\vec{A}+\hat{B})+(\vec{A}-\hat{B}) \quad \Rightarrow \vec{R} = 2\vec{A}$

Thus $\vec{R}$ and $\vec{A}$ are in same direction.

65. Projection = $\frac{(3\hat{i}+4\hat{k}).\hat{j}}{|\hat{j}|} = 0$

66. If θ is the angle between $\vec{A}_1$ and $\vec{A}_2$, then

$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos\theta$

or $3^2 = 2^2 + 3^2 + 2 \times 2 \times 3\cos\theta$

$\therefore \cos\theta = -1/3$

Now, $(\vec{A}_1 + 2\vec{A}_2)\,(3\vec{A}_1 - 4\vec{A}_2)$

$= 3\vec{A}_1.\vec{A}_1 + 6\vec{A}_2.\vec{A}_1 - 4\vec{A}_1.\vec{A}_2 - 8\vec{A}_2.\vec{A}_2$

$= 3 \times 2^2 + 2 \times 2 \times 3 \times (-1/3) - 8 \times 3^2 = -64$